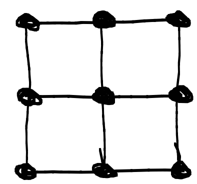
Quantum computation

Lectures 10 – 12

Cluster-state quantum computation

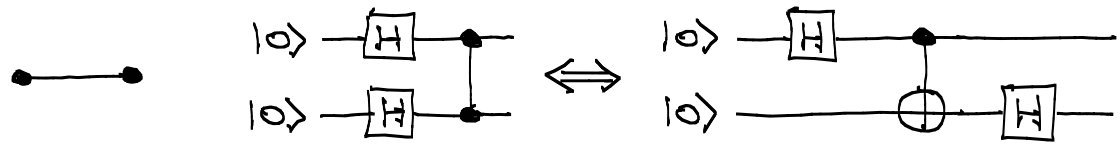
Cluster-state QC



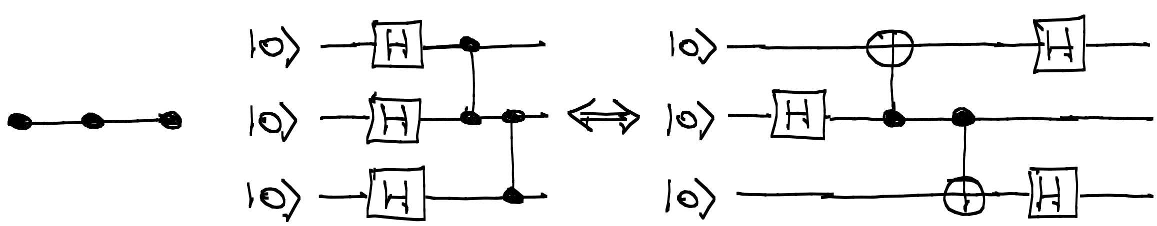
3x3 cluster

1. Qubits start in $|0\rangle$
2. Hadamard on each qubit (all qubits in $|\bar{0}\rangle = |+\rangle$)
3. C-SIGN between neighbors in graph

Examples

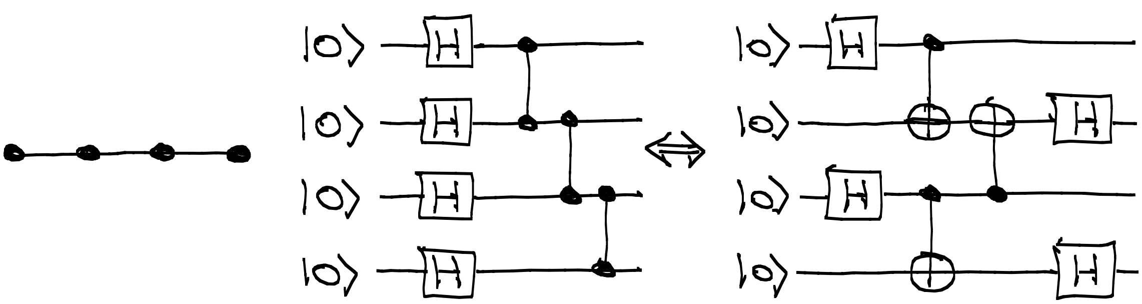


$$I\otimes H|\beta_{00}\rangle = \frac{1}{\sqrt{2}}\left(|0\bar{0}\rangle + |1\bar{1}\rangle\right)$$

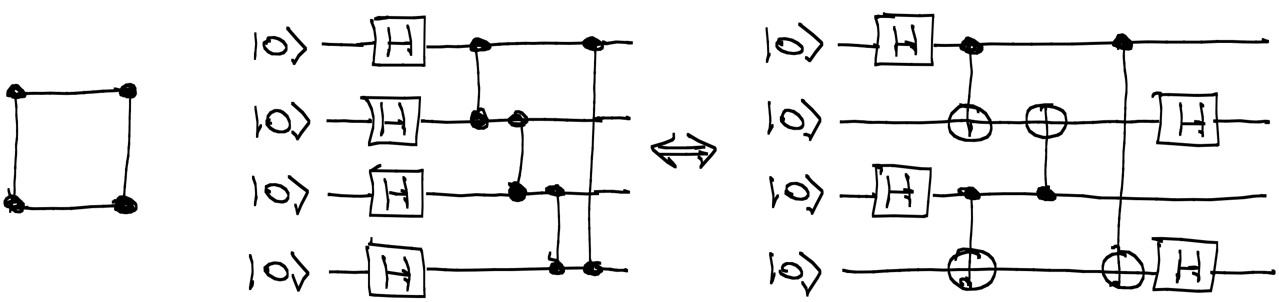


$$H\otimes I\otimes H|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|\bar{0}0\bar{0}\rangle + |\bar{1}1\bar{1}\rangle\right)$$

$$|GHZ\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$



$$\frac{1}{2}\left(|0\bar{0}0\bar{0}\rangle + |1\bar{1}0\bar{0}\rangle + |0\bar{1}1\bar{1}\rangle + |1\bar{0}1\bar{1}\rangle\right)$$



$$\frac{1}{2}\left(|0\bar{0}0\bar{0}\rangle + |1\bar{1}0\bar{1}\rangle + |0\bar{1}1\bar{1}\rangle + |1\bar{0}1\bar{0}\rangle\right)$$

Graph states

A graph state is a little more general than a cluster state. For any graph (nodes with edges), the corresponding graph state is obtained by starting all qubits (nodes) in the $|\bar{0}\rangle = |+\rangle$ state and applying a C-SIGN between all qubits connected by edges.

Making a graph state in one fell swoop

Hamiltonian for C-SIGN

$$H = -\frac{1}{4}(Z - I) \otimes (Z - I) \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \frac{1}{4}(-I\otimes I + Z\otimes I + I\otimes Z - Z\otimes Z)$$

$$\Lambda(Z) = e^{-iH\pi} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= e^{+i\pi/4}\left(e^{-iZ\pi/4} \otimes e^{-iZ\pi/4}\right) e^{iZ\otimes Z\pi/4}$$

For all the C-SIGNs in a graph state, we would use the Hamiltonian

$$H = -\frac{1}{4}\sum_{j<k} B_{jk}(Z_j - I_j)(Z_k - I_k)$$

Adjacency matrix: $B_{jk} = \begin{cases} 0, & j \text{ and } k \text{ unconnected} \\ 1, & j \text{ and } k \text{ connected} \end{cases}$

$$= -\frac{1}{8}\sum_{j,k} B_{jk}(Z_j - I_j)(Z_k - I_k)$$

$\mathcal{N}(j) = \begin{pmatrix}\text{neighborhood} \\ \text{of } j\end{pmatrix}$

$$= -\frac{1}{4}N_J + \frac{1}{4}\sum_j N_j Z_j - \frac{1}{4}\sum_{j<k} B_{jk} Z_j Z_k$$

$= \{k \mid B_{jk} = 1\}$

$N_j = |\mathcal{N}(j)| = \sum_k B_{jk}$

$$\left|\begin{matrix}\text{graph} \\ \text{state}\end{matrix}\right\rangle = e^{-iH\pi} \underbrace{|\bar{0}\rangle^{\otimes N}}_{\frac{1}{2^{N/2}}\sum_a |a\rangle} = \frac{1}{2^{N/2}}\sum_a |a\rangle (-1)^{\sum_{j<k} B_{jk} a_j a_k}$$

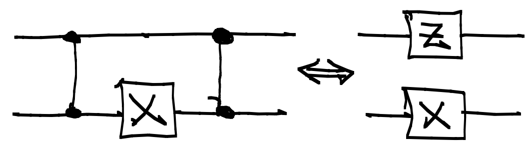
$N_J = \frac{1}{2}\sum_j N_j = \frac{1}{2}\sum_{j,k} B_{jk}$

Relation to the stabilizer formalism: A graph state is the unique simultaneous +1 eigenstate of the stabilizer generators

$$K_j = X_j \prod_{k \in \mathcal{N}(j)} Z_k \quad , \quad j = 1, \dots, N.$$

$$|0\rangle^{\otimes N} \xrightarrow{\text{Hadamards}} |\bar{0}\rangle^{\otimes N} \xrightarrow{\text{C-SIGNs}} \frac{1}{2^{N/2}} \sum_a |a\rangle (-1)^{\sum'_{j<k} a_j a_k}$$

$$Z_j \qquad\qquad X_j \qquad\qquad K_j = X_j \prod_{k \in \mathcal{N}(j)} Z_k$$

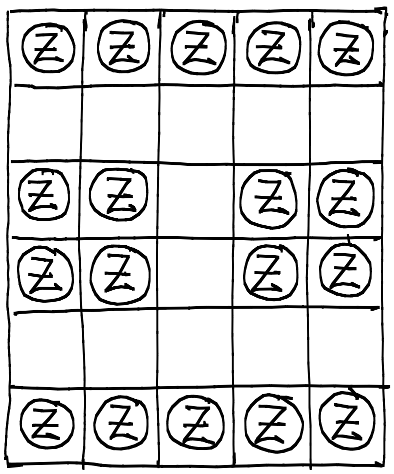


For a 2D cluster state, if there are three or more rows and columns, one can fold the grid into a torus, so that every qubit has four neighbors. The Hamiltonian becomes

$$H = -\frac{1}{2} N + \underbrace{\sum_j Z_j}_{\substack{\text{spins in a}\\ \text{uniform magnetic}\\ \text{field}}} - \frac{1}{4} \underbrace{\sum_{j<k} B_{jk} Z_j Z_k}_{\substack{\text{ferromagnetic}\\ \text{interaction}}}$$

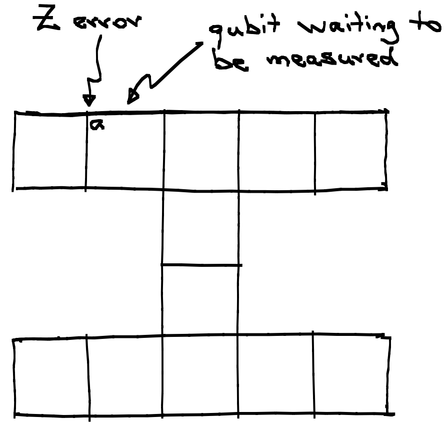
Now on to <u>cluster-state QC</u>

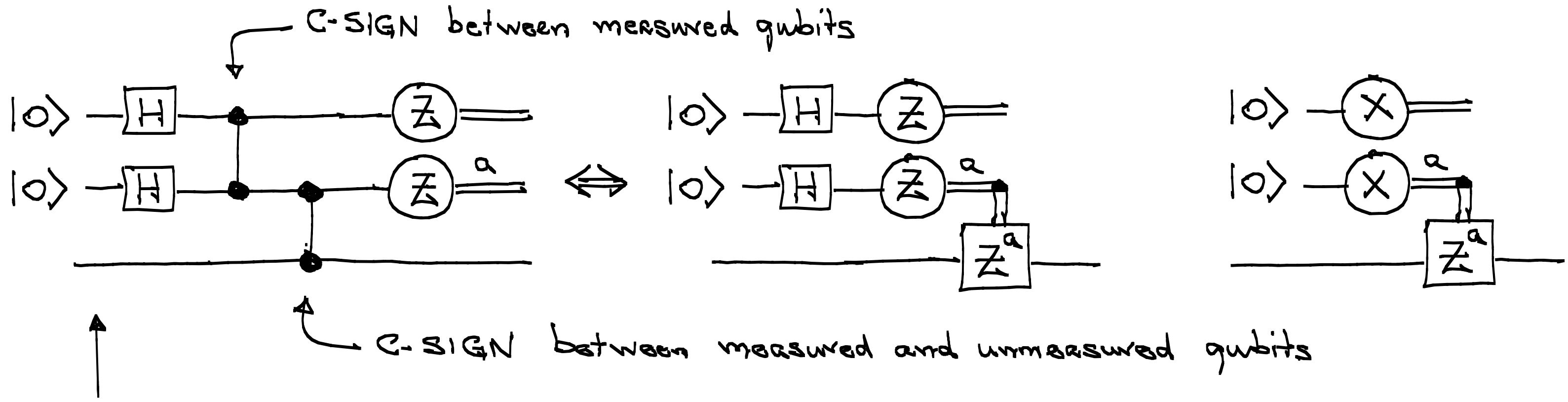
<u>Initialization</u>: make Z-measurements to lay out the circuit



After measurement, measured qubits are discarded. Remaining qubits have an initial Hadamard, C-SIGNs between remaining neighbors and Z errors.

Why not make this directly and avoid the Z errors?





There would be an $H|0\rangle$ here, but the identity doesn't depend on that input

<u>Single-qubit gates</u>

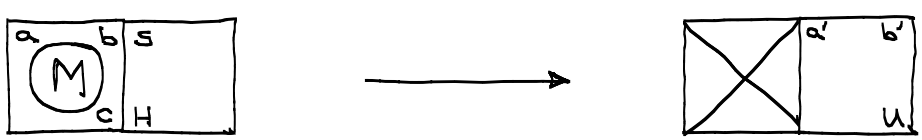
$$\theta_b = (-1)^b \theta$$

$$M = Z^\dagger(\theta_b)\, X\, Z(\theta_b)$$

$$U = H\, Z(\theta)$$

$$a' = b + s$$

$$b' = a + c$$



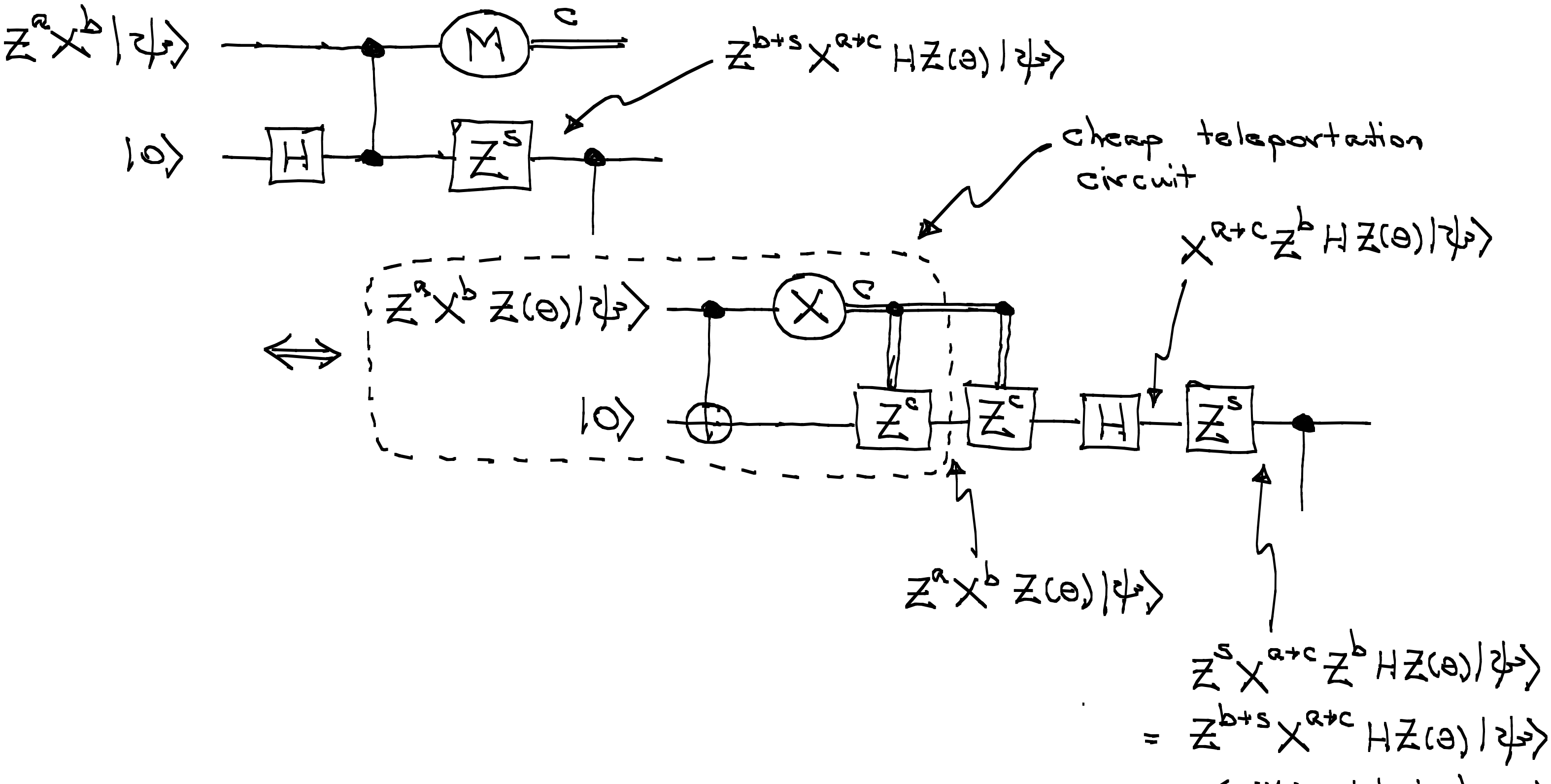
qubit in state $|\psi\rangle$, no initial Hadamard, $Z^a X^b$ Pauli errors, C-SIGN to right, and then measurement of M with result c

qubit in $|0\rangle$ with initial Hadamard, initial Z^s Pauli error, C-SIGNs to left and to further qubits in circuit

discarded qubit

qubit in state $U|\psi\rangle$ (no initial Hadamard), $Z^{a'} X^{b'}$ Pauli errors, C-SIGNs to further qubits in circuit

$$M = Z^\dagger(\theta_b)\, X\, Z(\theta_b) \qquad Z(\alpha) = e^{-iZ\alpha/2}$$
$$\theta_b = (-1)^b\,\theta$$



① Measuring X applies a Hadamard gate.

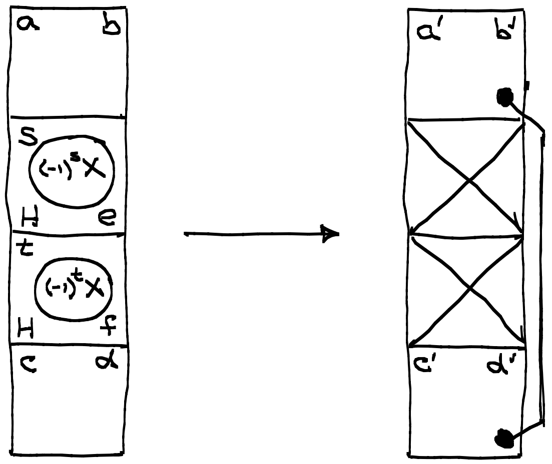
② Arbitrary $U = \underbrace{Z(\alpha)\, X(\beta)\, Z(\gamma)}_{\text{Z-X (Euler) decomposition}} = H\ HZ(\alpha)\ HZ(\beta)\ HZ(\gamma)$

$Z(\theta) = H\ HZ(\theta)$

③ Only need measurements in the equatorial plane of the Bloch sphere

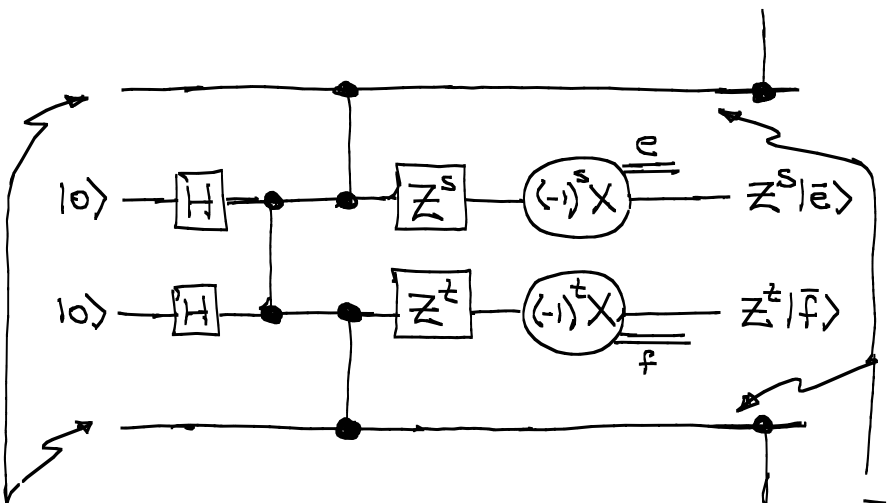
④ The measurements on the leftmost qubits have the initial Hadamard, so that the unitary performed on the neighboring qubit is $HZ(\theta)H = X(\theta)$. One can live with this, e.g., by setting $\theta = 0$ and starting the circuit on the second leftmost qubit, or one could leave the Hs off the leftmost qubits in the preparation of the cluster state (edge effect).

Two-qubit gates: C-SIGN



$a' = a+d+f$
$b' = b$

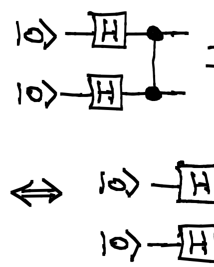
$c' = c+b+e$
$d' = d$



$Z^a X^b \otimes Z^c X^d |\Psi\rangle$

$Z^{a+d+f} X^b \otimes Z^{c+b+e} X^d \Lambda(Z)|\Psi\rangle$

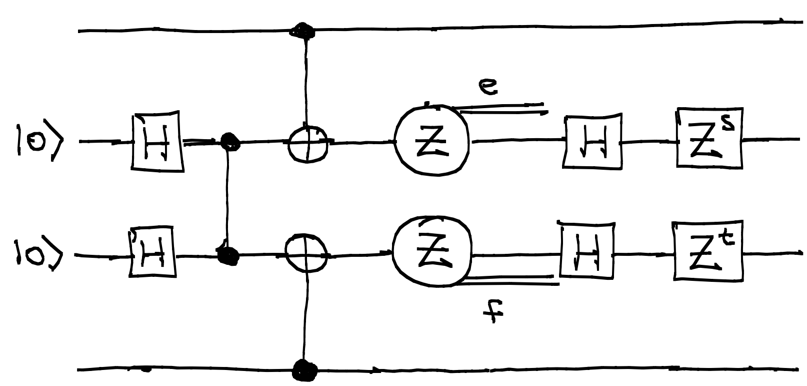
+1 eigenstate of $H \otimes H$



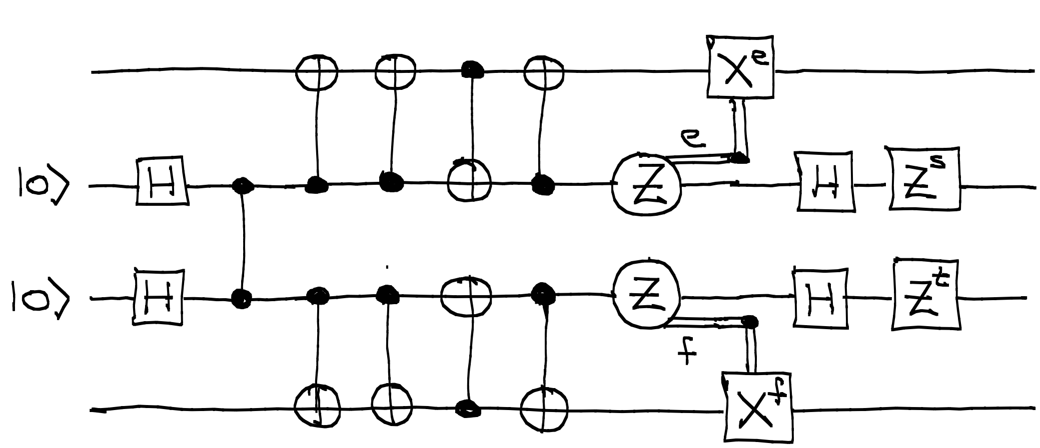
$I \otimes H|\beta_{00}\rangle = H \otimes I|\beta_{00}\rangle$

$I \otimes H|\beta_{00}\rangle$

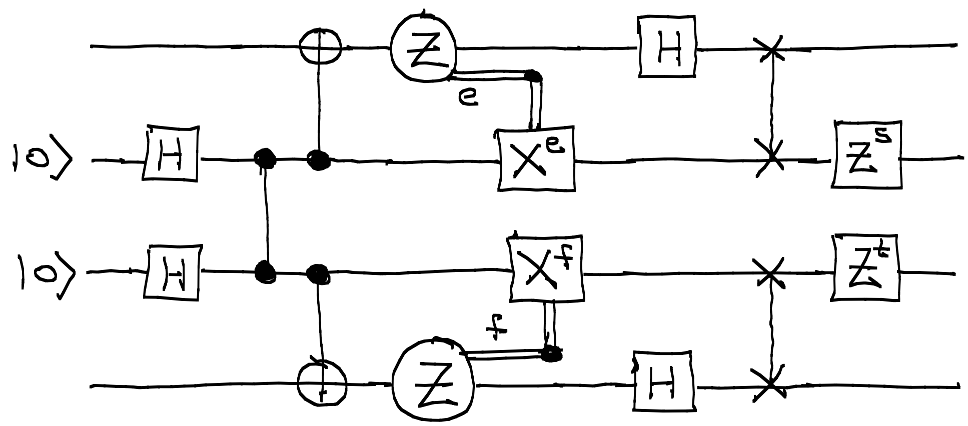
$\Leftrightarrow$



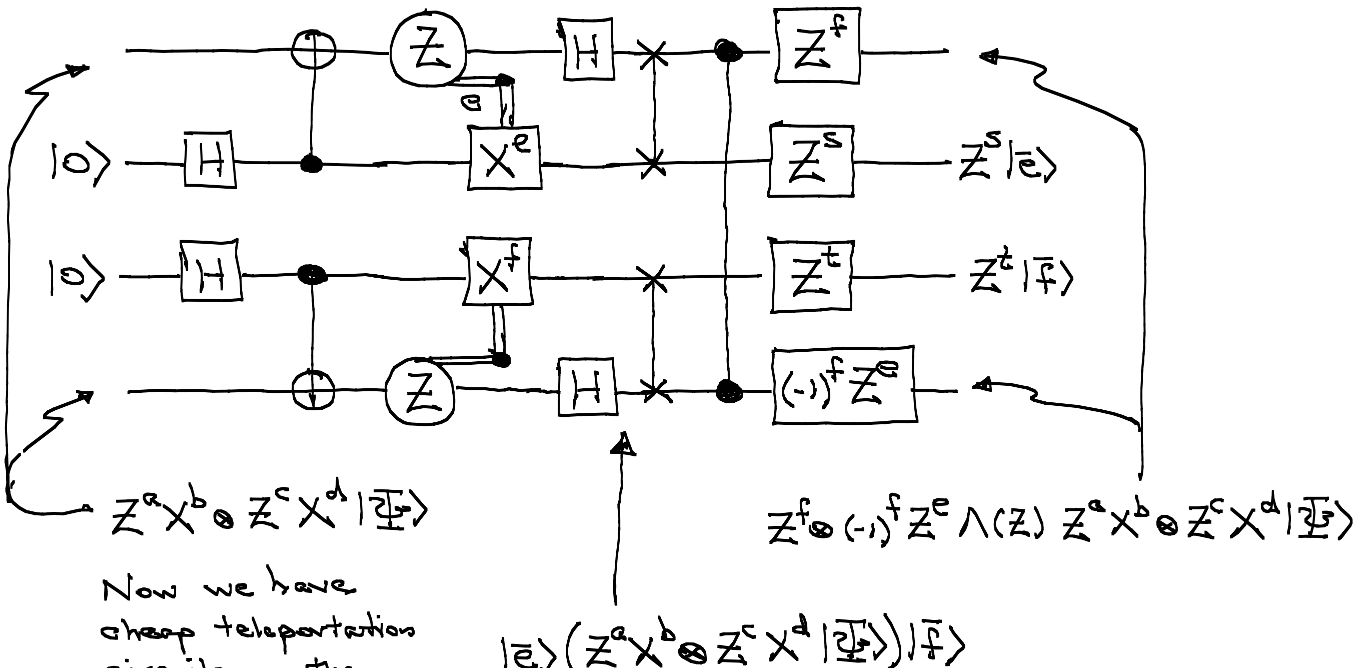
$\Leftrightarrow$



$\Longleftrightarrow$



$\Longleftrightarrow$

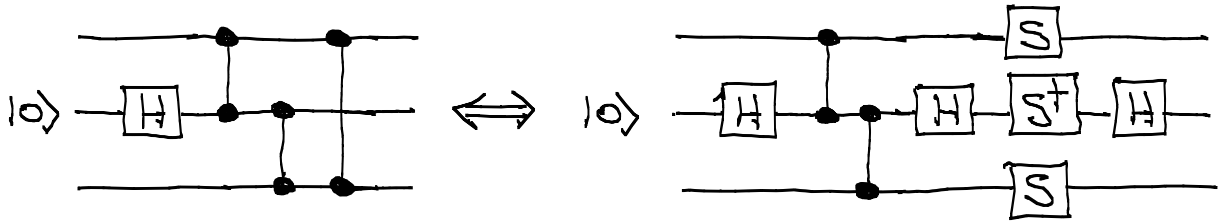


Now we have cheap teleportation circuits on the upper and lower pairs of qubits.

<u>Final measurement:</u> Measure in standard basis; Z errors are irrelevant, and X errors flip results.

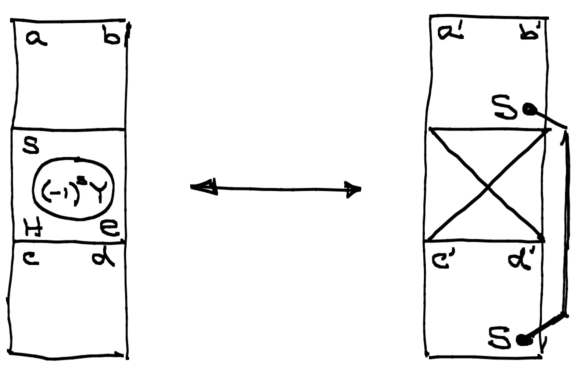
Another approach

One can make the cluster-state approach work with one row of buffering Z measurements between active rows, instead of two. This introduces more correlated Pauli errors between active qubits, but that really doesn't matter. The single-qubit gates are implemented just as before (and the final measurements work the same way), but we need a new implementation for the C-SIGN. We will need the following circuit identity, which is due to Bryan Eastin.



Proof on appended pages

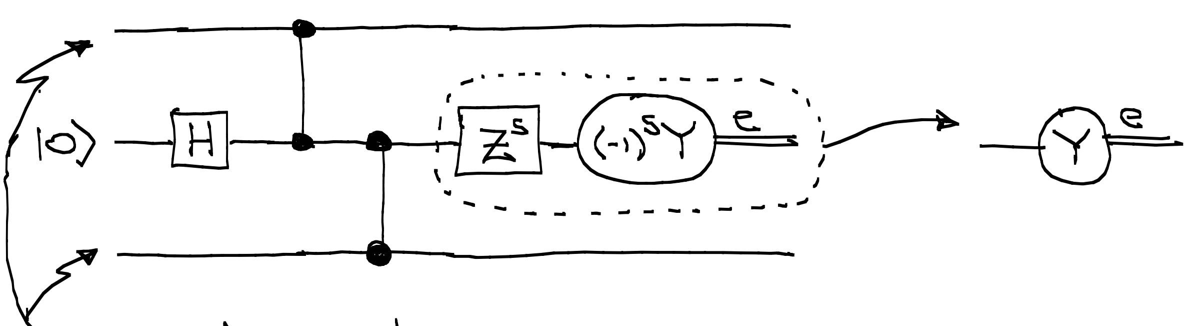
Here's what we actually show:



$a' = a + b + d + e$
$b' = b$

$c' = c + b + d + e$
$d' = d$

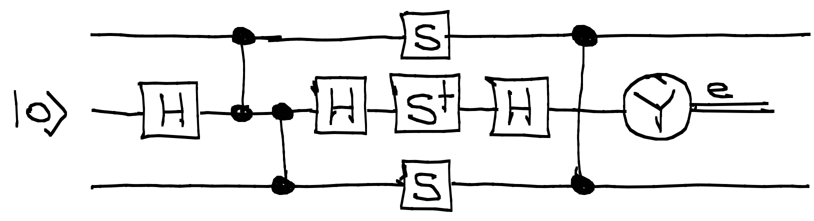
This is an example of the identities used to show the local equivalence of graph states.



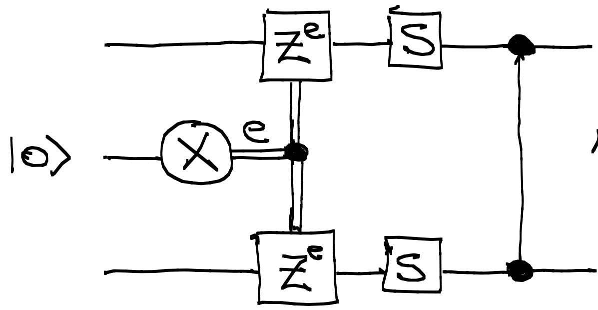
$Z^a X^b \otimes Z^c X^d |\Phi\rangle$

$\Longleftrightarrow$

Eastin identity



$\Leftrightarrow$

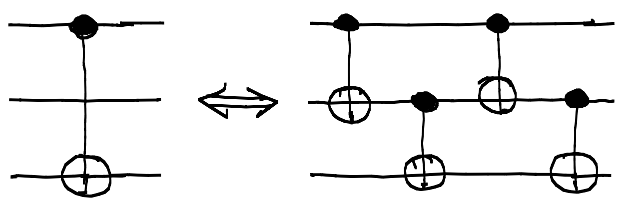


$$\Lambda(Z) S Z^{a+e} X^b \otimes S Z^{c+e} X^d |\Psi\rangle$$
$$= \Lambda(Z)\, Z^{a+e} Y^b \otimes Z^{c+e} Y^d\, S \otimes S |\Psi\rangle$$
$$= \Lambda(Z)\, Z^{a+b+e} X^b \otimes Z^{c+d+e} X^d\, S \otimes S |\Psi\rangle$$
$$= Z^{a+b+d+e} X^b \otimes Z^{c+b+d+e} X^d \Lambda(Z) S \otimes S |\Psi\rangle$$

global phases ignored

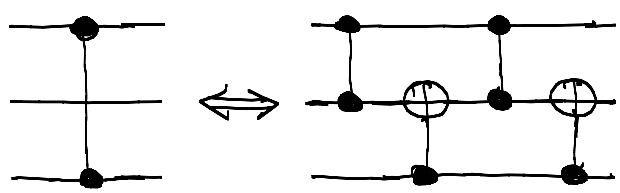
Proof of Eastin circuit identity

Start with

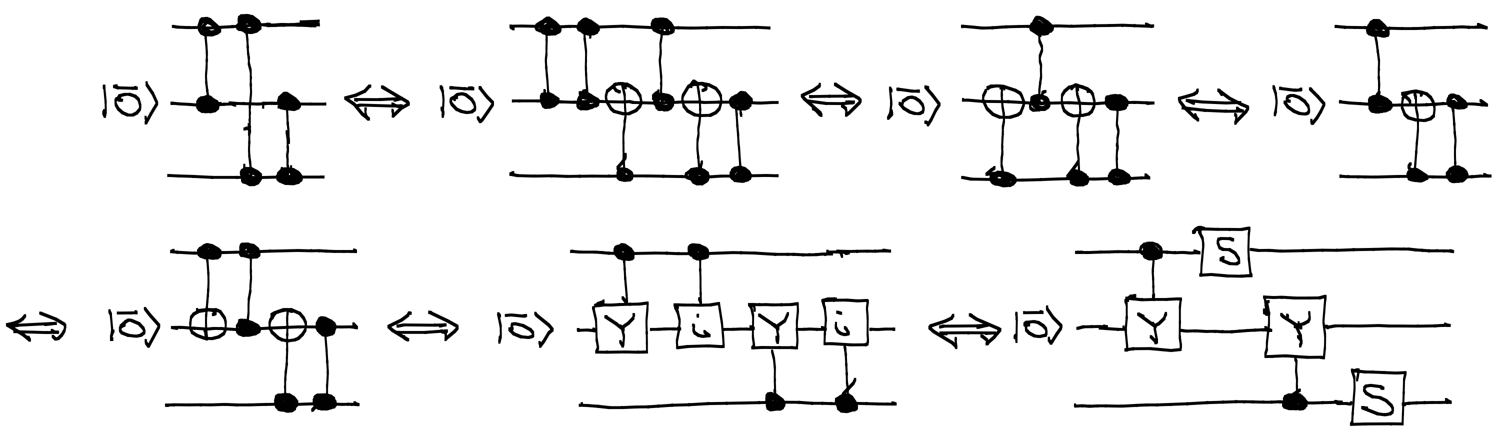


$$|a,b,c\rangle \to |a, a\oplus b, c\rangle$$
$$\to |a, a\oplus b, a\oplus b\oplus c\rangle$$
$$\to |a, b, a\oplus b\oplus c\rangle$$
$$\to |a, b, a\oplus c\rangle$$

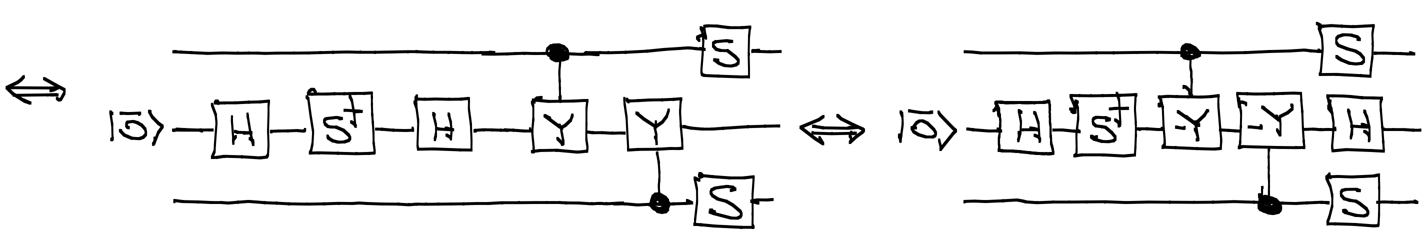
Convert this to C-SIGN by putting Hadamards at the beginning and end of the second and third wires.



Now we get serious.



$H S^{\dagger} H |\bar{0}\rangle = |\bar{0}\rangle$



B

$SYS^\dagger = -X$