Quantum computation

Lectures 19 - 20a

From classical error correction to Shor's 9-qubit code

Why quantum error correction?

Incorrect operations introduce errors.

Decoherence: info encoded in correlations is destroyed by local errors.

Example: bit flip on a cat state

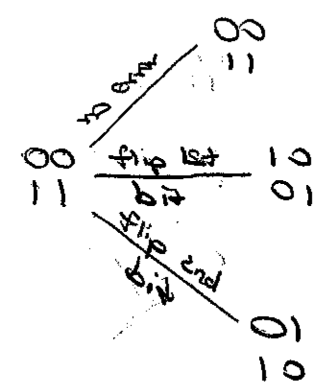
Continuous dissipation and decay

Contrast classical computer where errors are eliminated passively by cooling within a large potential well that accommodates error.

Classical error correction

Info encoded as 0, 1

Error: bit flip

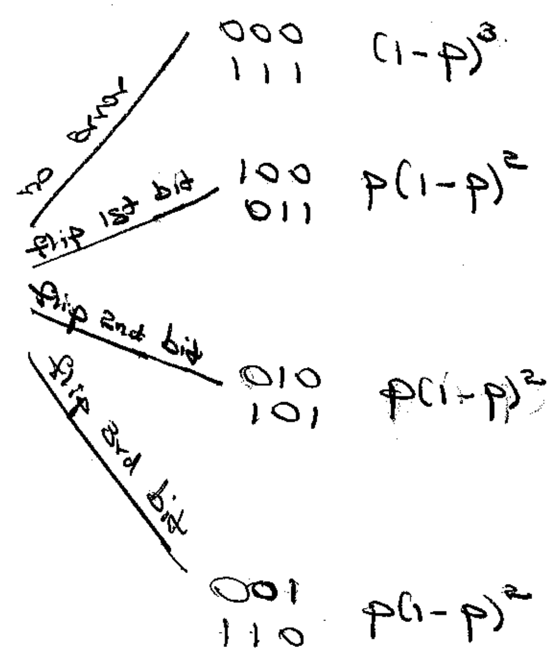


Detection of error, but not correction

Redundancy: repetition code

$0_L = 000$

$1_L = 111$



Majority vote locates error. Offending qubit is flipped to correct error.

All 8 bit strings are realized by single-bit errors, so can't correct 2- and 3-bit errors

Win if $p > 3p^2(1-p) + p^3 = 3p^2 - 2p^3$

$$\Leftrightarrow \quad 0 < 2p^3 - 3p^2 + p = p(1-2p)(1-p)$$

$$\Leftrightarrow \quad p < 1/2$$

Problems for qec:

1. No cloning — how get redundancy
2. Measurements destroy superpositions
3. Other errors

   Continuous errors from decoherence and incorrect operations

   Phase flips

Bit-flip quantum code

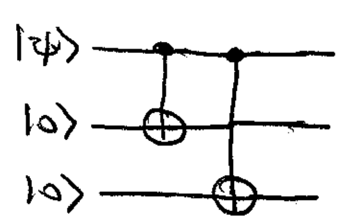
$|0_L\rangle = |000\rangle$

$|1_L\rangle = |111\rangle$

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \rightarrow |\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$

Code subspace is the 2d-subspace spanned by $|0_L\rangle$ and $|1_L\rangle$.

Encoding



Cloning doesn't come up. Redundancy is replaced by storing information nonlocally.

$|\psi\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$
$= \alpha|000\rangle + \beta|111\rangle$

- no error: $\alpha|000\rangle + \beta|111\rangle$
- flip 1st qubit: $\alpha|100\rangle + \beta|011\rangle$
- flip 2nd qubit: $\alpha|010\rangle + \beta|101\rangle$
- flip 3rd qubit: $\alpha|001\rangle + \beta|110\rangle$

4 ⊥ 2d subspaces exhaust 8d space.

Detect and locate single-qubit errors by measuring which of the 4 2d-subspaces the 3 qubits are in. By measuring only the 2 bits to identify the subspace and not the 3 bits to identify a basis state, the measurement does not destroy superpositions.

Syndrome measurement

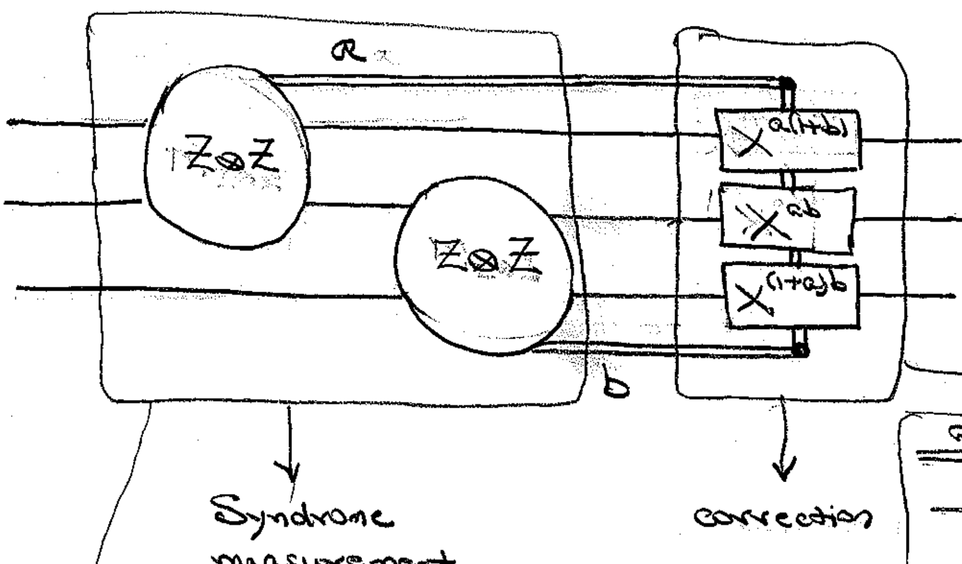
Measure $Z \otimes Z \otimes I$ and $I \otimes Z \otimes Z$

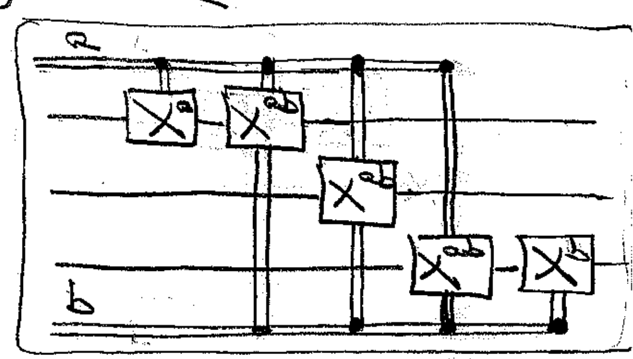
ZZI $\quad$ IZZ

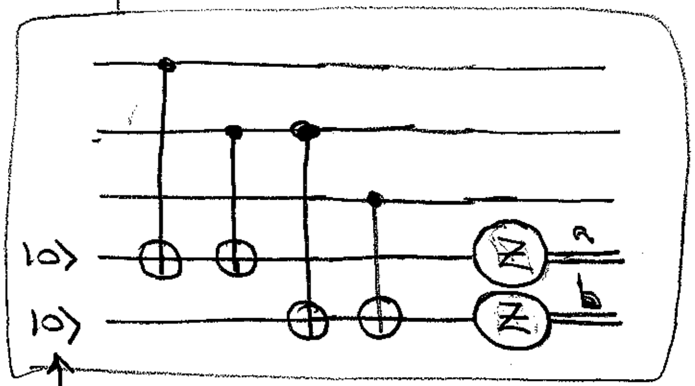
$Z_1 Z_2$ $\quad$ $Z_2 Z_3$

More compact notations

a	b	
0	0	no error
1	0	1st qubit flipped
1	1	2nd qubit flipped
0	1	3rd qubit flipped

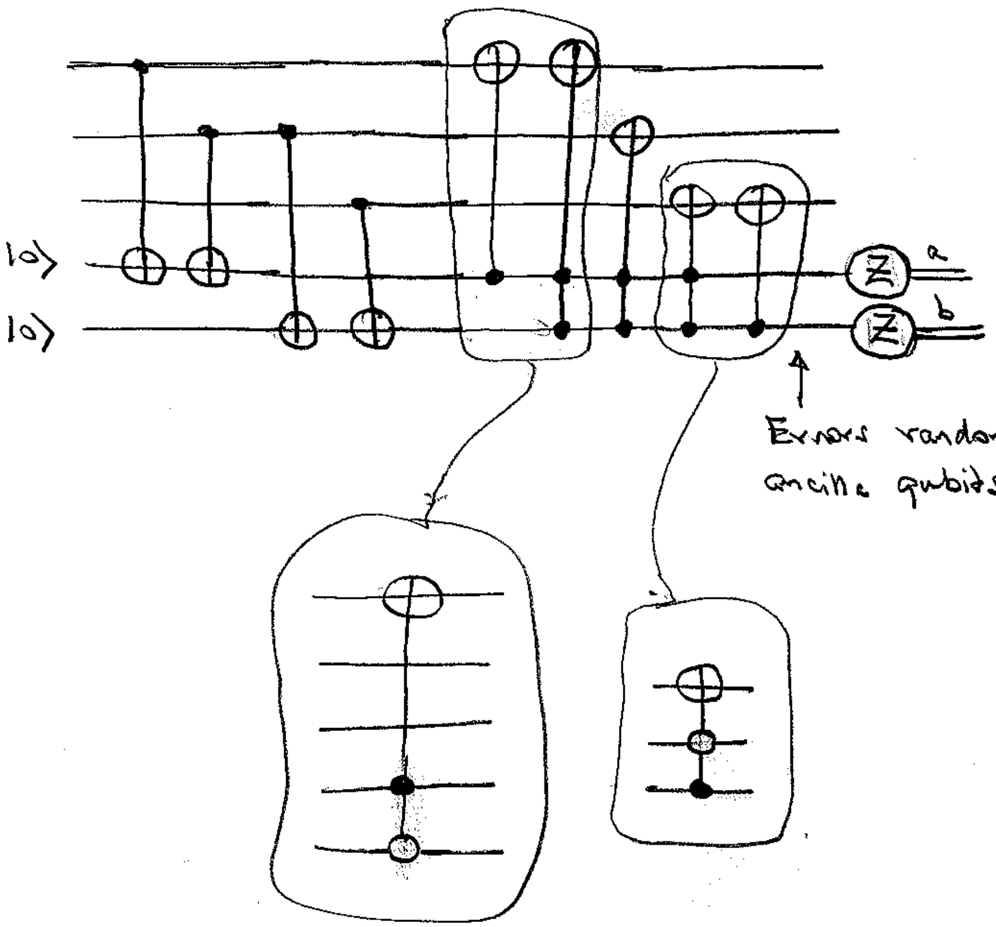






ancilla qubits

Coherent error correction circuit

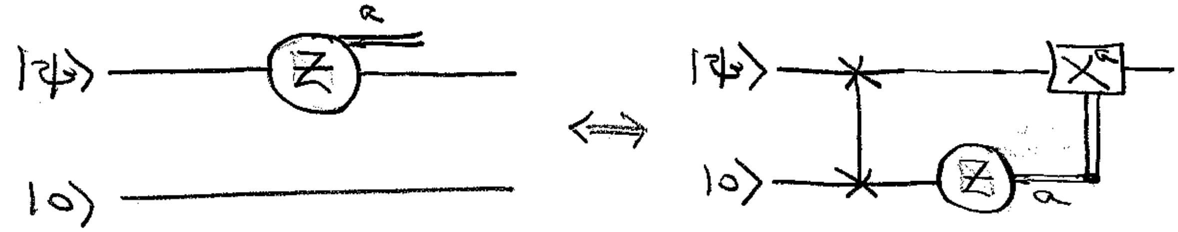


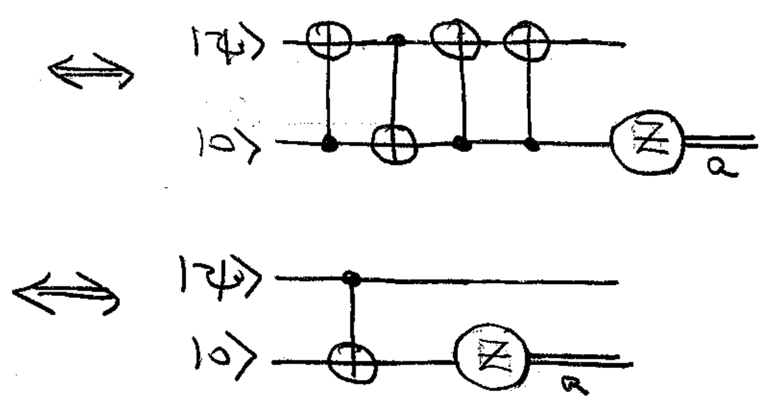
Measurements reveal what error was corrected. Generally omitted.

Continuous errors: Error is $e^{-iX_1\epsilon/2}$. Syndrome and correction force the system to commit either to no error or a bit flip.

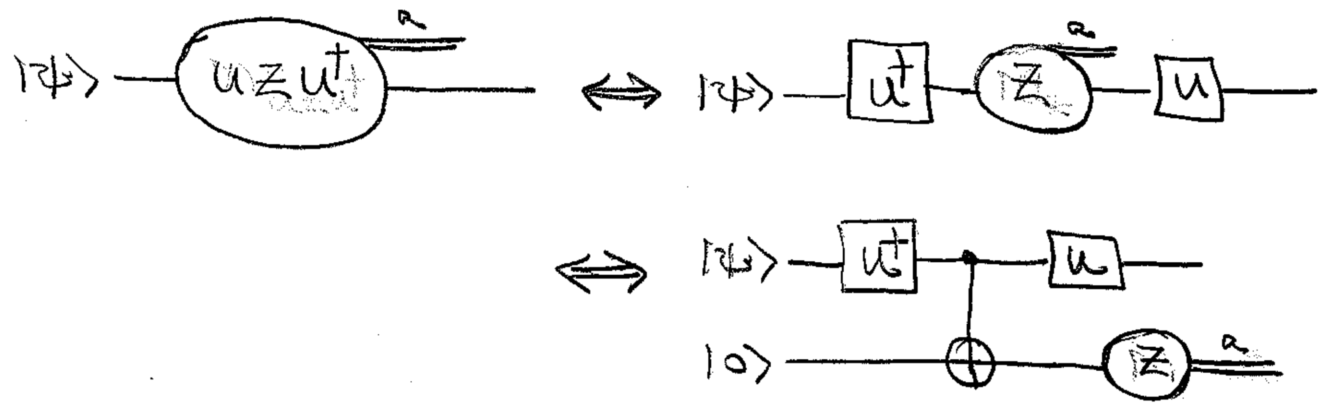
Error correction digitizes quantum computation, although in the coherent version, this digitization is never realized.

A final digression on measurements and circuits

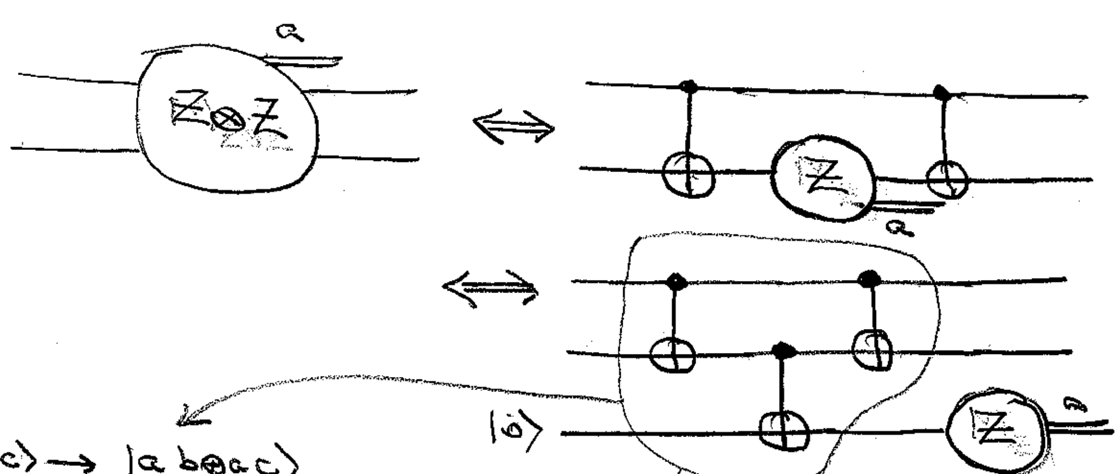




Turning a direct measurement into a measurement model



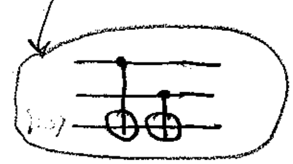
On to 2 qubits



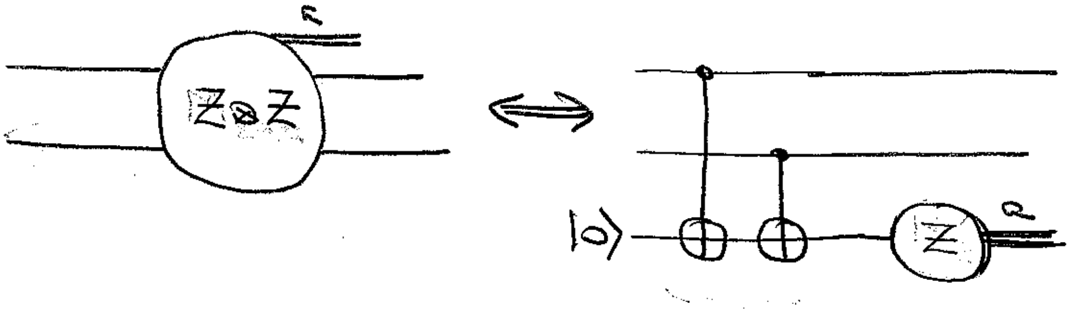
$|abc\rangle \longrightarrow |a, b\oplus a, c\rangle$

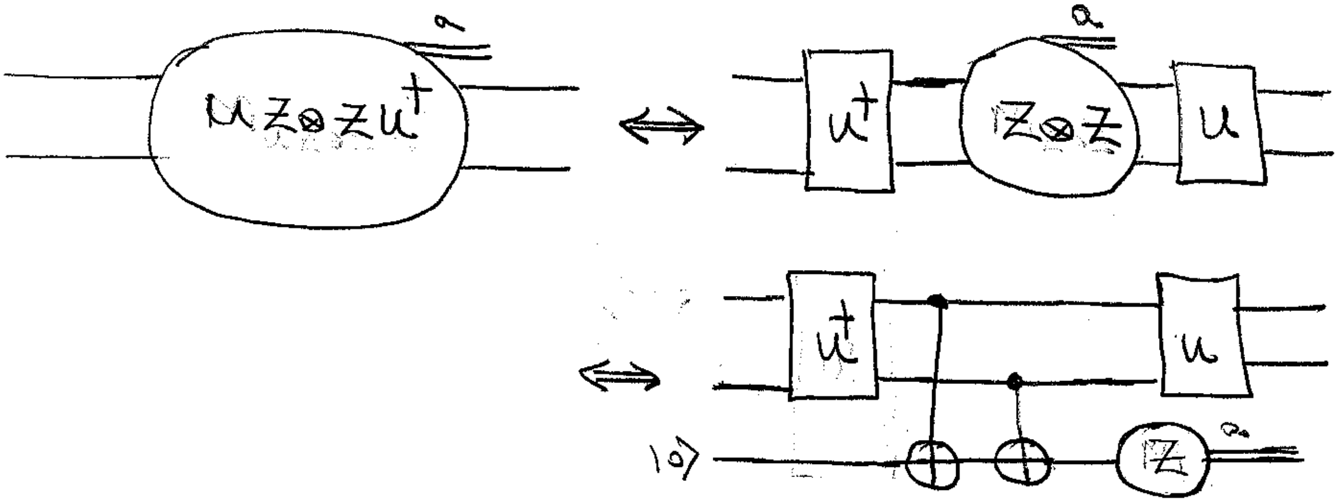
$\longrightarrow |a, b\oplus a, c\oplus b\oplus a\rangle$

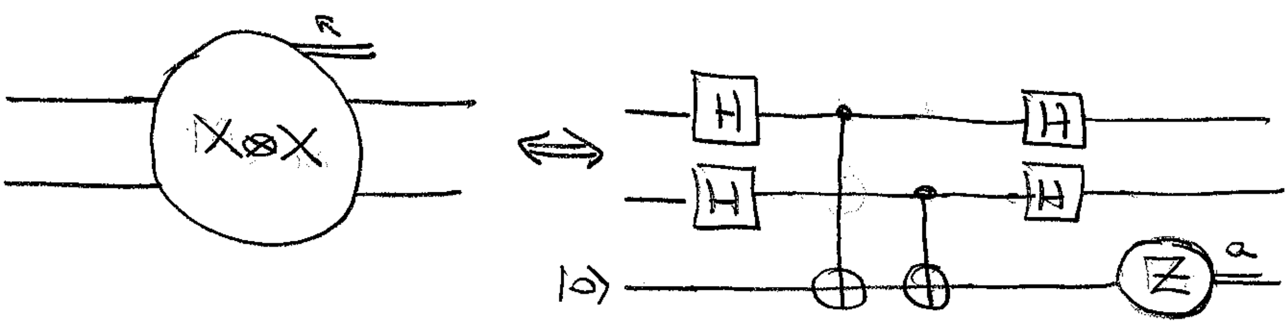
$\longrightarrow |a, b, c\oplus b\oplus a\rangle$



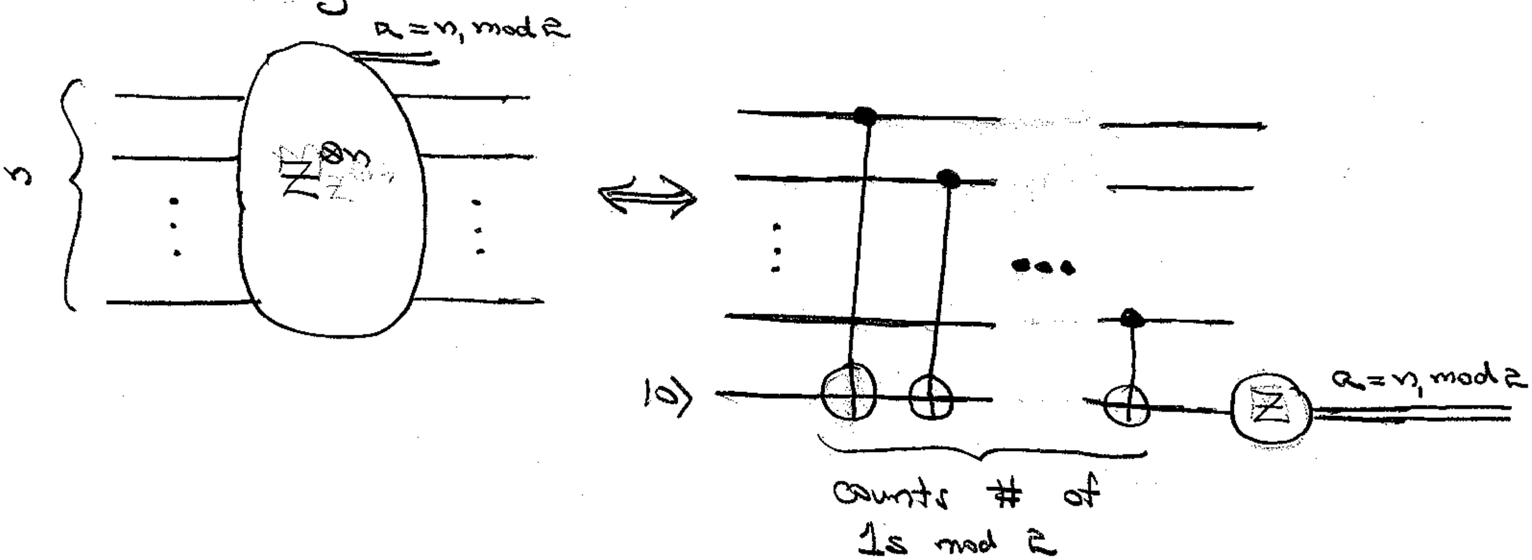
B







Generalizing



Phase flips

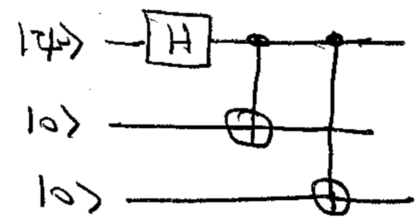
We have to do something more (or different) because bit flips on 3 qubits already fill up the entire Hilbert space.

① The bit-flip code works just as well regardless of which basis in the code subspace we use for $|0_L\rangle$ and $|1_L\rangle$. For example, we could use

$$|0_L\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|1_L\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)$$

Encoding



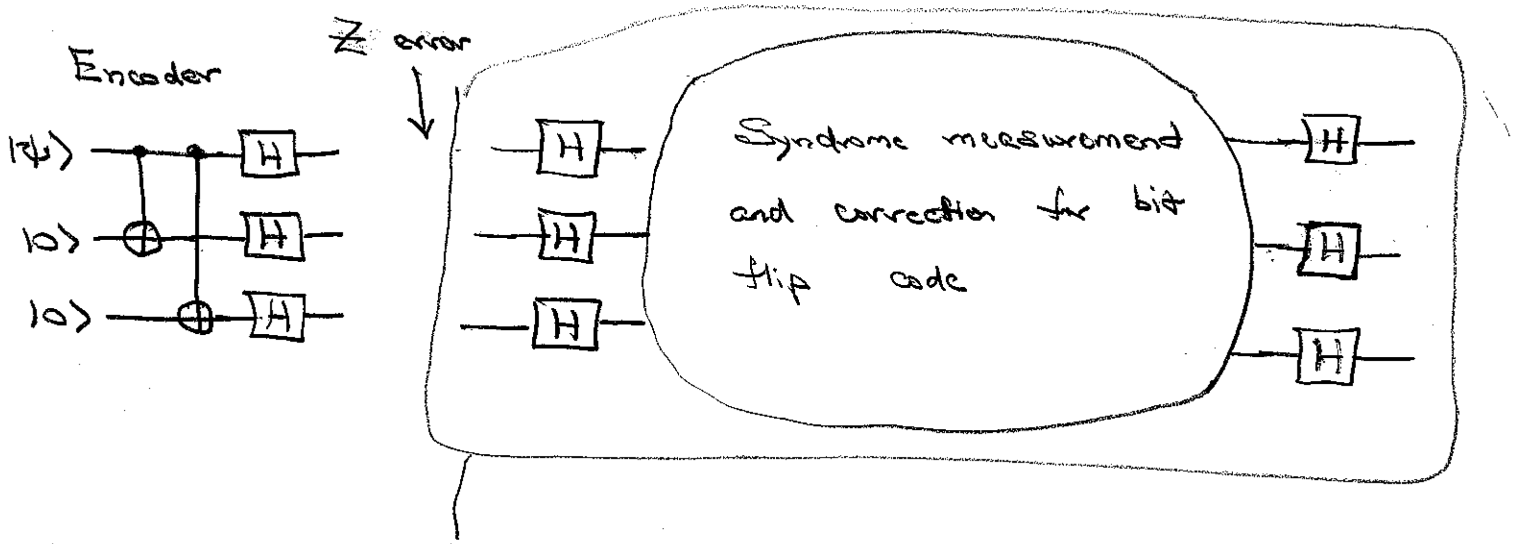
$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \longrightarrow |\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle$$

② Phase-flip code

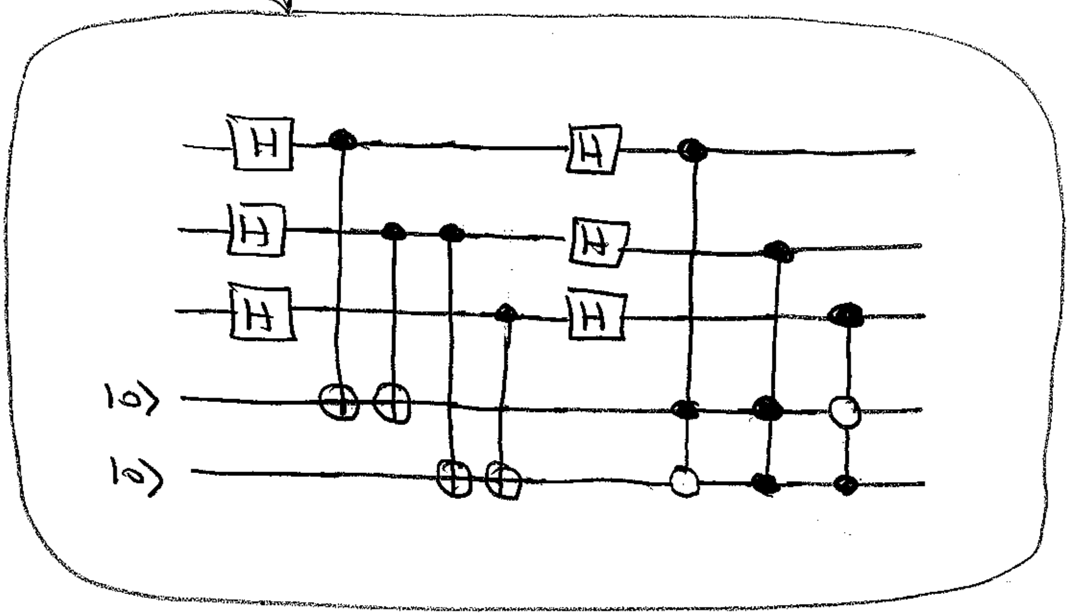
A phase flip is a bit flip in the $|\pm\rangle$ basis, so simply transform the bit-flip code to this basis

$$|0_L\rangle = |\bar{0}\bar{0}\bar{0}\rangle$$

$$|1_L\rangle = |\bar{1}\bar{1}\bar{1}\rangle$$



Syndrome measurement: X_1X_2 and X_2X_3
Correction uses C-SIGNS instead of C-NOTs.



Getting it all right: Shor's 9-bit code

$$|+\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right)$$

$$|-\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle - |111\rangle\right)$$

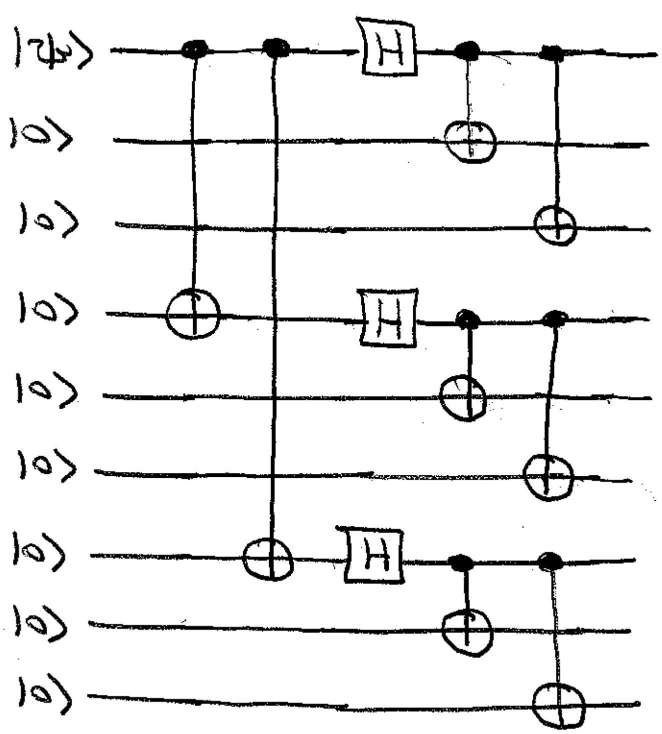
Z on any qubit flips $|\pm\rangle$. $X_1X_2X_3$ is the analogue of X on a single qubit because

$$X_1X_2X_3|\pm\rangle = \pm|\pm\rangle$$

$$|0_L\rangle = |+++\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)\left(|000\rangle + |111\rangle\right)$$

$$|1_L\rangle = |---\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)\left(|000\rangle - |111\rangle\right)$$

Encoding



Single-qubit X errors

Syndrome measurements

Z_1Z_2, Z_2Z_3, Z_4Z_5, Z_5Z_6, Z_7Z_8, Z_8Z_9

Recovery:

$X_1, \ldots, X_9$ based on syndrome

Single-qubit Z errors

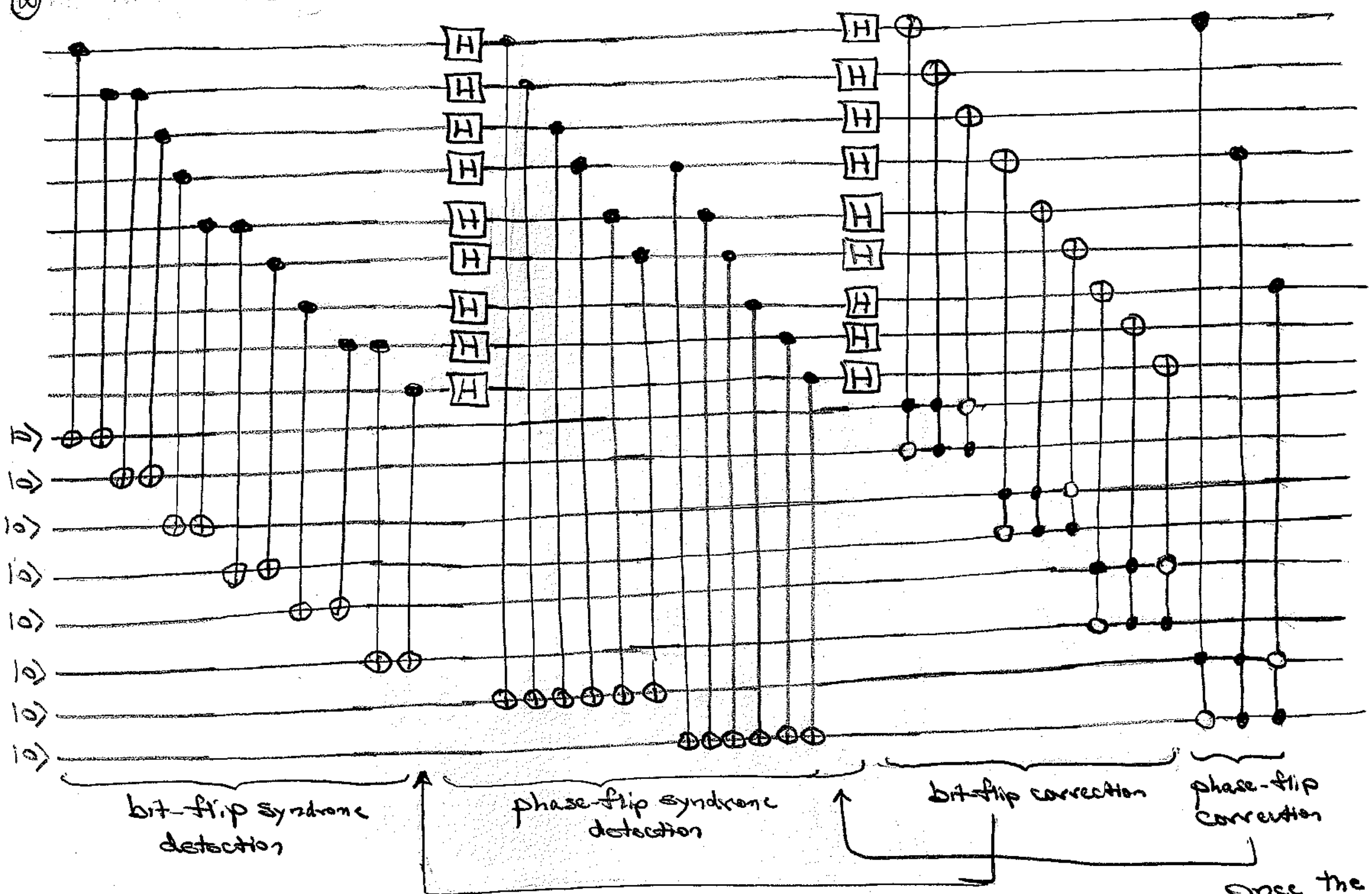
Syndrome measurements

$X_1X_2X_3X_4X_5X_6$,

$X_4X_5X_6X_7X_8X_9$

Recovery

Z_1, Z_4, Z_7 based on syndrome



This correction circuit for bit flips can be moved over here, right after bit-flip syndrome detection; ie, we can do first all the bit-flip detection and correction, and then do the phase-flip detection and correction.

Once the bit-flip correction is moved, the phase-flip correction can be moved before the Hadamards by changing the multi-controlled Z's to multi-controlled-NOTs.

Counting: No error $\quad$ X errors $\quad$ Y errors $\quad$ Z errors

No error	X errors	Y errors	Z errors
1	9	9	9

but only 3 distinct syndromes (degenerate code)

$\longrightarrow$ 22 $\perp$ 2d subspaces to accommodates the errors.

There are $2^9/2 = 256$ 2d subspaces available. Shor's code can be used to correct some multi-qubit errors, and these fill out the entire ($2^9 = 512$)-dimensional Hilbert space (see Appendix).

Quantum Hamming bound (for single-qubit errors):

$$\underset{\substack{\uparrow \\ \text{no error}}}{2(1} + \underbrace{r}_{\text{\# of single-qubit errors / qubit}} N) \leq 2^N \iff r \leq \frac{2^{N-1} - 1}{N}$$

$N=3$, $r \leq 1$
$N=4$, $r \leq 7/4$
$N=5$, $\underline{\underline{r \leq 3}}$

Any single-qubit error can be written as a linear combination of I, X, Y, and Z, so correcting X, Y, and Z is sufficient. Syndrome detection commits the system to I, X, Y, or Z.

Questions:

1. How does error correction work generally? Mapping to $\perp$ subspaces?
2. Can we always correct by correcting an appropriate set of discrete errors?
3. How do we construct codes?

Ⓐ

Counting the Shor errors

It is clear from the circuit diagram that the Shor code can correct the following errors:

$$\begin{pmatrix}\text{no error or}\\ X \text{ on any qubit}\\ \text{in first block}\end{pmatrix} \text{ and } \begin{pmatrix}\text{no error or}\\ X \text{ on any qubit}\\ \text{in second block}\end{pmatrix} \text{ and } \begin{pmatrix}\text{no error or}\\ X \text{ on any qubit}\\ \text{in third block}\end{pmatrix}$$

$$\text{and } \begin{pmatrix}\text{no error or}\\ Z \text{ on any of}\\ \text{the nine qubits}\end{pmatrix}$$

$$4 \times 4 \times 4 \times 10 = 640 \text{ errors}$$

But in counting subspaces, we must take into account the fact that the Z errors in each block are indistinguishable, so

$$4 \times 4 \times 4 \times (1 + \underset{\substack{\uparrow\\ Z \text{ errors in the 3 blocks}}}{3}) = 2^8 = 256 \text{ orthogonal subspaces}$$