Quantum computation

Lectures 7–9

Measurement-based quantum computation

Circuit model

Initial state $|00\ldots0\rangle$

Ancilla qubits all in $|0\rangle$

One-qubit gates and C-NOT (or C-SIGN)

Measurement in computational basis

Universal initial state $|\psi\rangle$

One- and two-qubit measurements plus feedforward to subsequent measurements

Measurement-based QC: Teleportation QC TQC
Cluster-state QC CSQC (1WQC)

Review

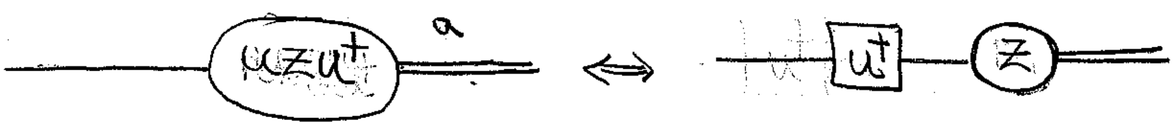
(1) Measurement in computational basis

$$\textcircled{Z} = \textcircled{M} = \textcircled{M_Z}$$

actual result is $(-1)^a$

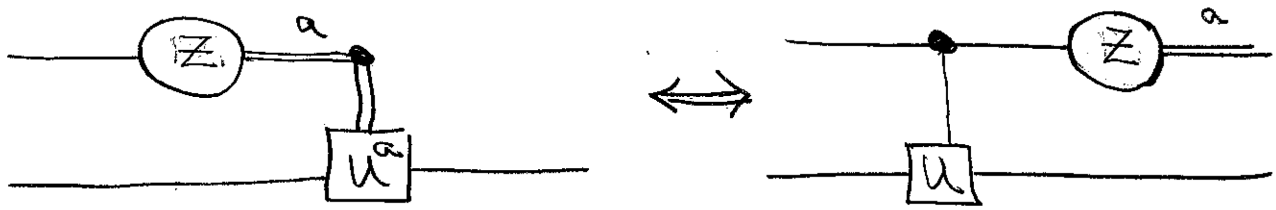
$p_a = |\langle a|\psi\rangle|^2$

(2) Measurement in another basis $U|a\rangle$



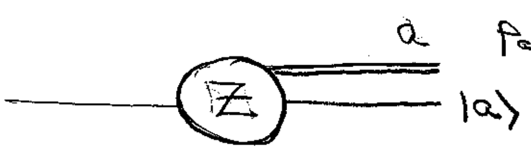
$$(UZU^\dagger)\,U|a\rangle = UZ|a\rangle = (-1)^a\,U|a\rangle$$

③ Classical vs. quantum control



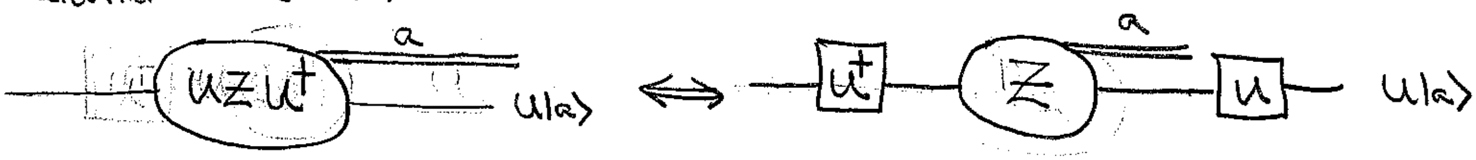
Generalize to keeping measured qubits

① Measurement in computational basis



$p_a = |\langle a|\psi\rangle|^2$

② Measurement in another basis $U|a\rangle$



relative-state decomposition

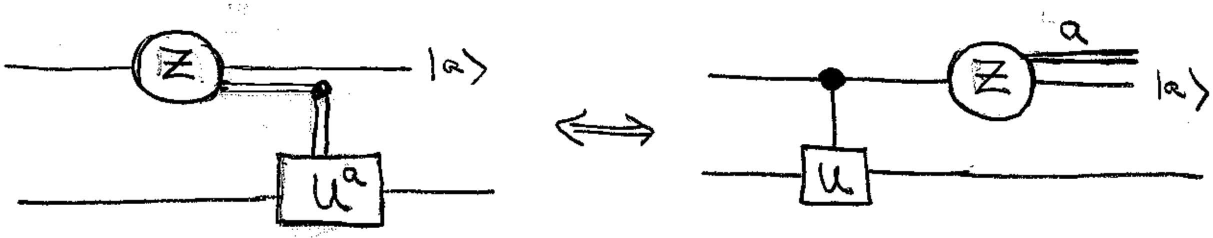
rest of world $\downarrow$

$$|\psi\rangle = \sum_a c_a U|a\rangle \otimes |\psi_a\rangle \longrightarrow U|a\rangle \otimes |\psi_a\rangle$$

$$\downarrow \qquad p_a = |c_a|^2 \qquad \uparrow$$

$$\sum_a c_a |a\rangle \otimes |\psi_a\rangle \longrightarrow |a\rangle \otimes |\psi_a\rangle$$

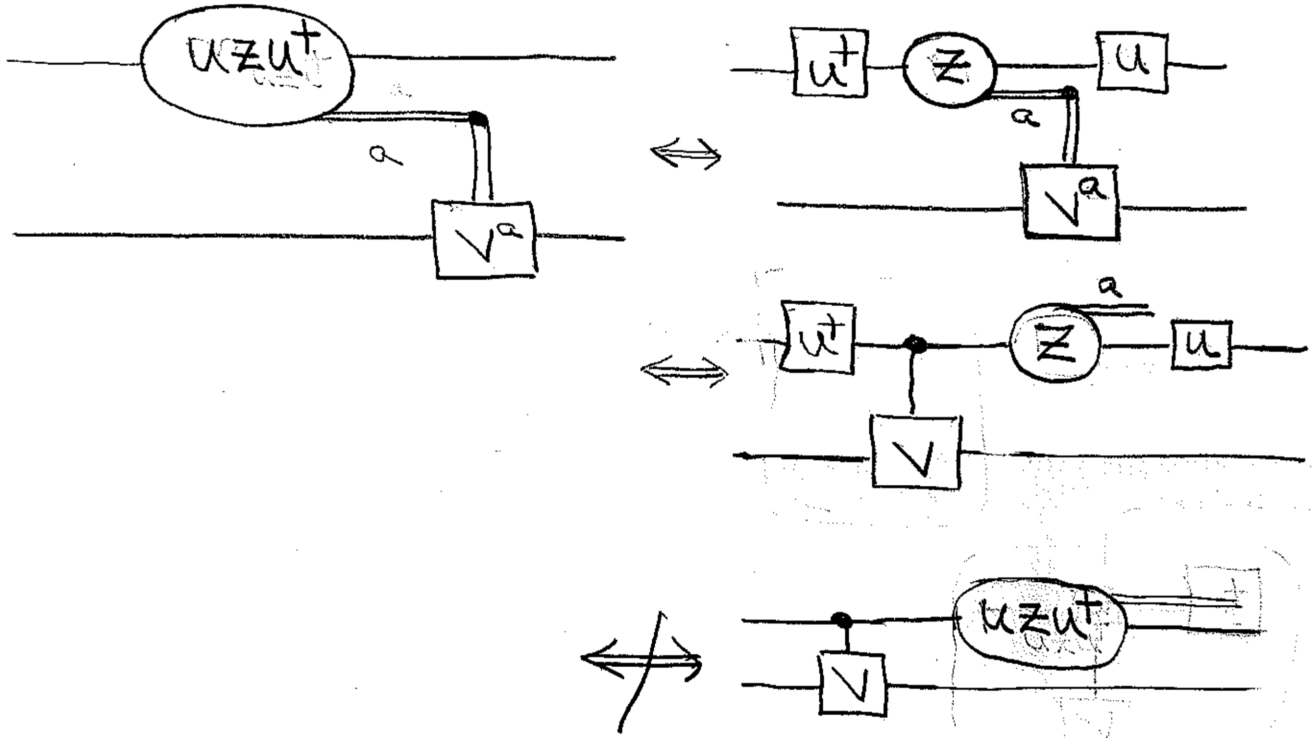
3 Classical vs. quantum control



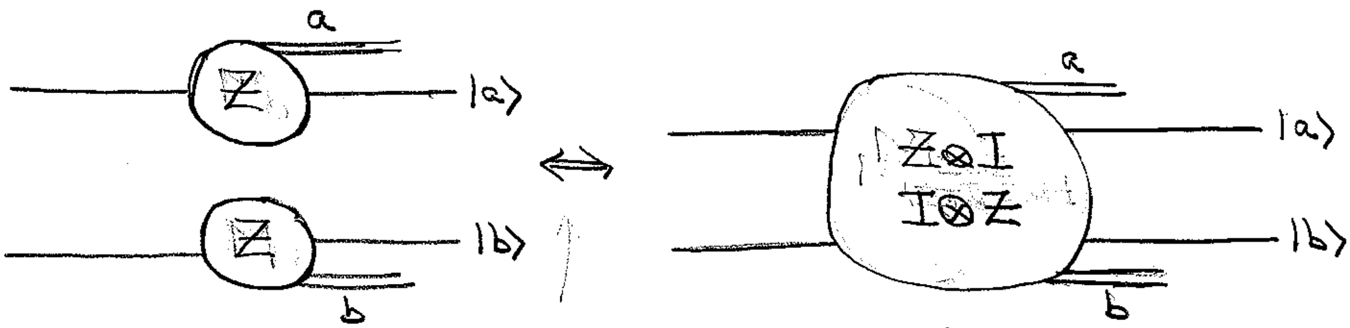
$$|\psi\rangle = \sum_{a,b} c_{ab}\, |a,b\rangle \otimes |\psi_{ab}\rangle \longrightarrow |a\rangle \otimes \sum_b c_{ab}\, |b\rangle \otimes |\psi_{ab}\rangle \longrightarrow |a\rangle \otimes \sum_b c_{ab}\, U^a |b\rangle \otimes |\psi_{ab}\rangle$$

$$\searrow \quad \sum_{a,b} c_{ab}\, |a\rangle \otimes U^a |b\rangle \otimes |\psi_{ab}\rangle \quad \nearrow$$

Putting 2 and 3 together

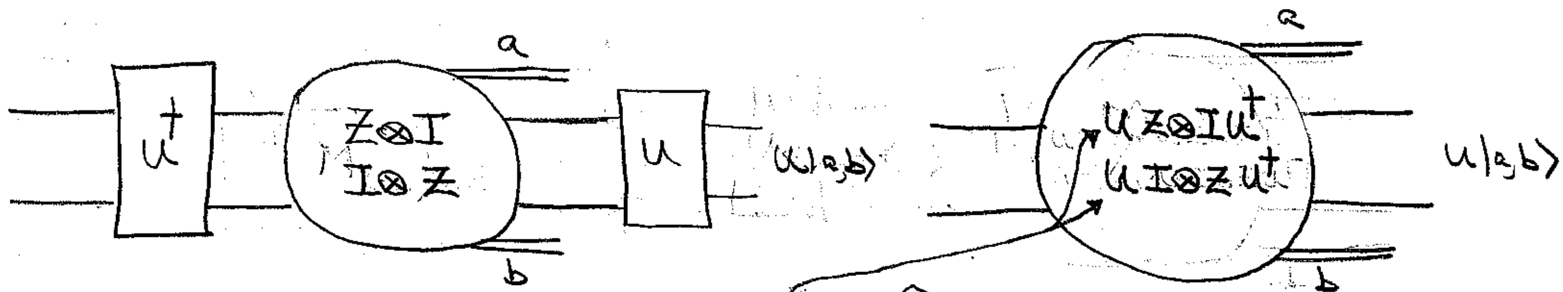


Two-qubit measurements



Measurement in computational basis

Same rule for measurements in another basis $U|a,b\rangle$



these two commute

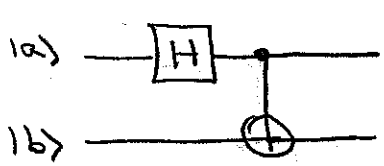
① $U = V\otimes W \Rightarrow$ 1-qubit measurements in bases $V|a\rangle$ and $W|b\rangle$.

② U not a tensor product

<u>Example</u>

$U = \Lambda(X)\, H\otimes I$

Bell basis



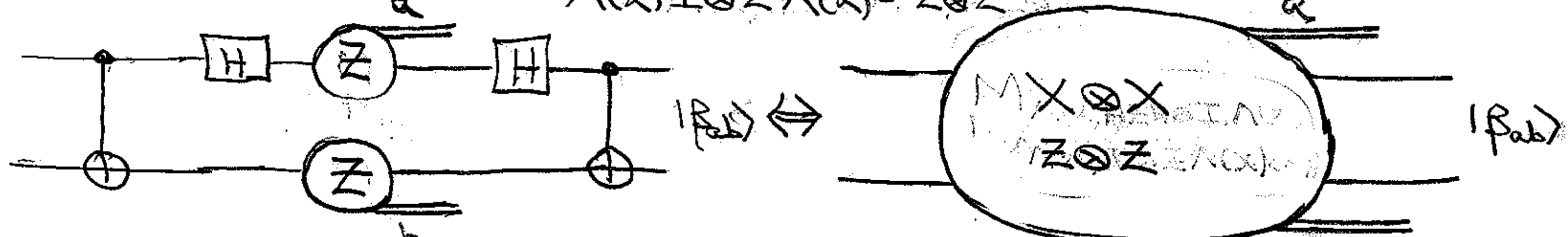
$$|\beta_{ab}\rangle = \frac{1}{\sqrt{2}}\sum_c (-1)^{ac}|c, b\oplus c\rangle = \frac{1}{\sqrt{2}}\left(|0b\rangle + (-1)^a|1\bar{b}\rangle\right)$$

$$= Z^a\otimes X^b|\beta_{00}\rangle = Z^aX^b\otimes I|\beta_{00}\rangle = I\otimes X^bZ^a|\beta_{00}\rangle$$

$$\Lambda(X)HZH\otimes I\,\Lambda(X) = X\otimes X$$

$$\Lambda(X)\, I\otimes Z\,\Lambda(X) = Z\otimes Z$$

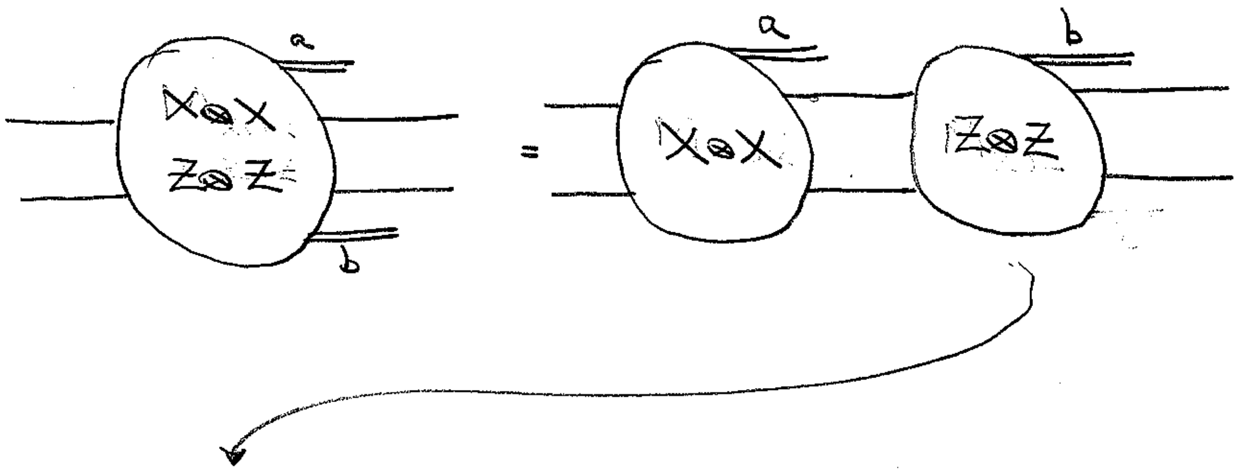
phase bit

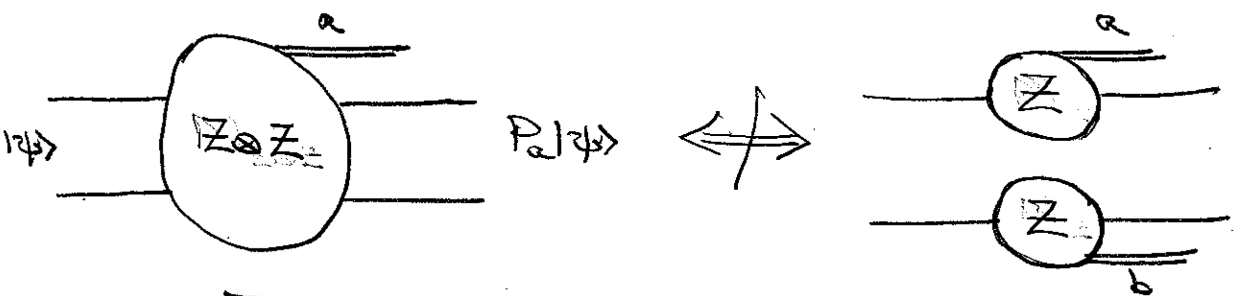


parity bit

Notation: $\Lambda(U)$  $V(U)$ 

We can also take a two-qubit measurement apart into two successive incomplete quantum measurements.





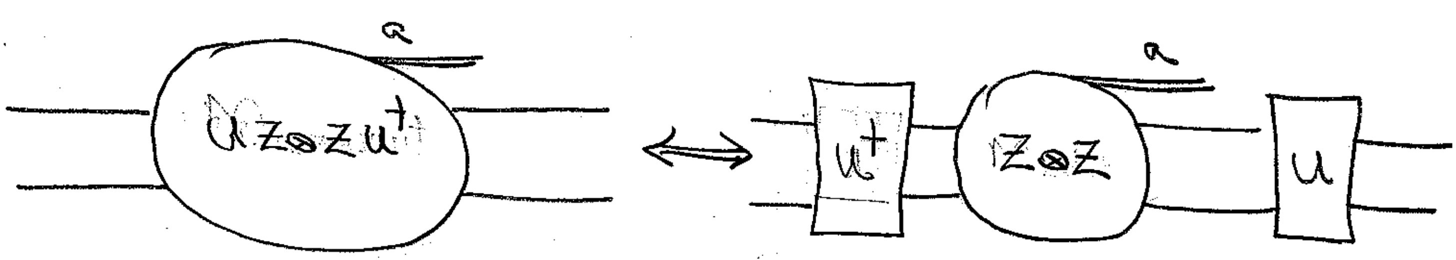
$$P_a = \sum_b |b, b\oplus a\rangle\langle b, b\oplus a|$$

$$= \sum_b |b\oplus a, b\rangle\langle b\oplus a, b|$$

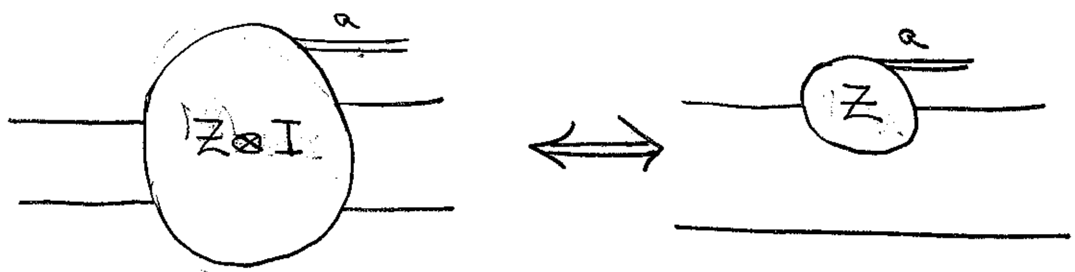
← measurement of parity

$$|\psi\rangle = \sum_{c,d} c_{cd}|c,d\rangle \longrightarrow \sum_b c_{b,b\oplus a}|b, b\oplus a\rangle$$

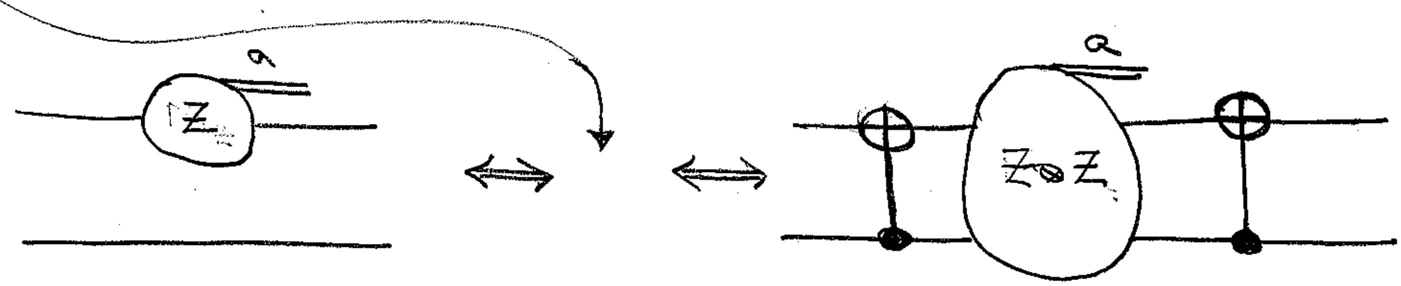
Same old rule to get to other bases:



Relation of incomplete 2-qubit measurement to 1-qubit measurement



Example: $Z\otimes I = V(X)\, Z\otimes Z\, V^{\dagger}(X)$



Pause to check:

$$|\psi\rangle = \sum_{c,b} c_{cb}|c,b\rangle \longrightarrow |a\rangle \otimes \sum_{b} c_{ab}|b\rangle$$

$$\sum_{c,b} c_{cb}\,\underbrace{|c\oplus b, b\rangle}_{\substack{\text{parity of}\\ \text{this state}\\ \text{is } c}} \xrightarrow[\text{Apply } P_a]{} \sum_{b} c_{ab}|a\oplus b, b\rangle$$

measurement of parity determines $c = a$

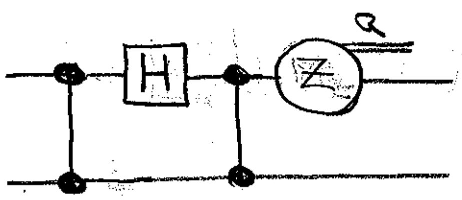
Tools for converting a circuit to measurements:

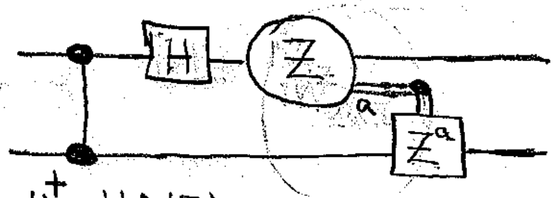
1. Replacing state preparation on a to-be-active wire with measurements.
2. Adding measurements on to-be-discarded qubit.
3. Moving unitaries through measurements by unitarily transforming the measurement basis.
4. Discarding terminal unitaries on a to-be-discarded qubit.
5. Changing quantum controls to classical controls through Z-measurements

This is the only way to eliminate the 2-qubit gates that involve to-be-active qubits.

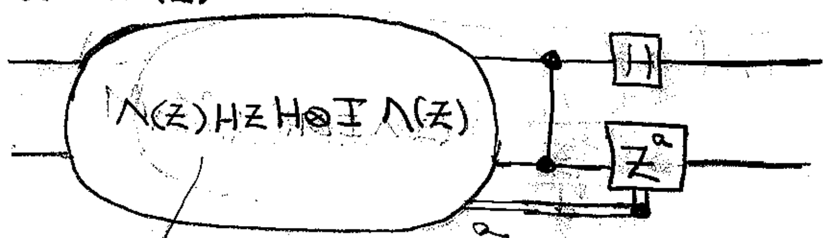
Example: 

Using output only on 2nd wire, convert to measurements.

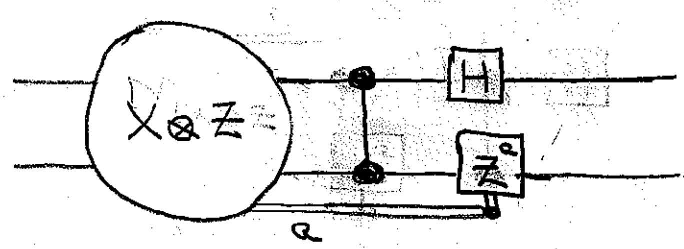


$\Longleftrightarrow$ 

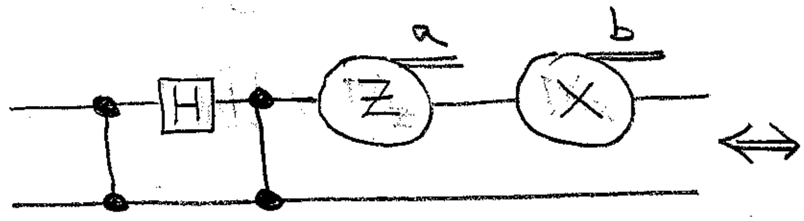
$U^{\dagger} = H\Lambda(Z)$

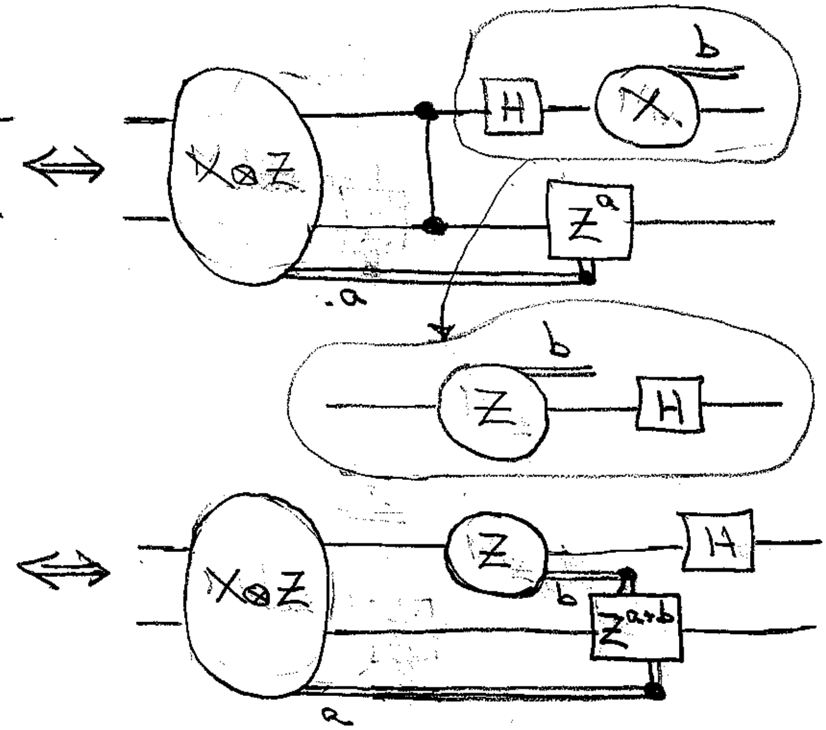
$\Longleftrightarrow$ 

$= \Lambda(Z)\, X\otimes I\, \Lambda(Z) = X\otimes Z$

$\Longleftrightarrow$ 

Need another measurement to eat up the C-PHASE:

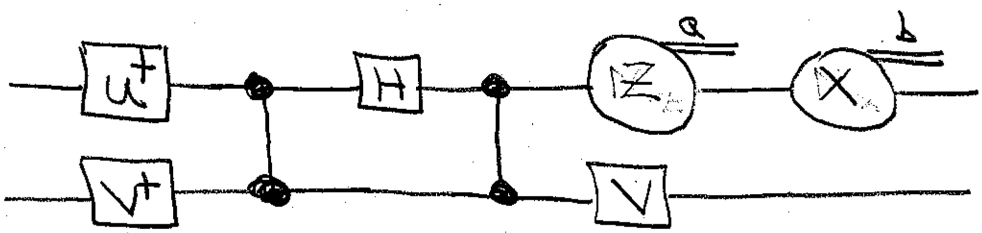


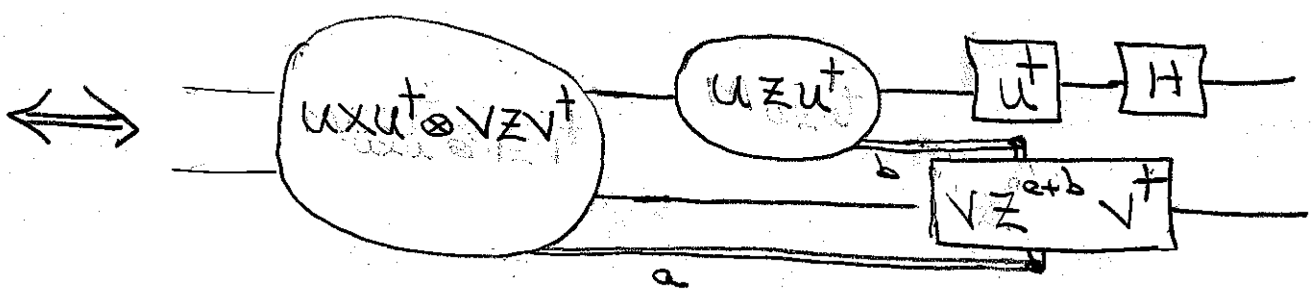


Can this be right? How can random result b in first circuit be of any use in the second?

Check on next page

Elaborating:





<u>Note:</u> The projections at measurements can be accompanied by arbitrary global phase changes. Not only are such phase changes physically meaningless, they are arbitrary and meaningless in the <u>algebra</u> of tracking states through measurements.

Check of circuit on p. 8:

Notation

$$|\bar{a}\rangle \equiv H|a\rangle \qquad \bar{a} \neq 1 \oplus a$$

$$|\bar{0}\rangle = H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$$

$$|\bar{1}\rangle = H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = |-\rangle$$

$$|\bar{a}\rangle = H|a\rangle = \sum_b (-1)^{ab}|b\rangle$$

$$|\psi\rangle = \sum_{b,c} c_{bc}|\bar{b},c\rangle \xrightarrow{\ \bullet\!\!-\!\!\bullet\ } \sum_{b,c} c_{bc}|\overline{b\oplus c},c\rangle \xrightarrow{H\otimes I} \sum_{b,c} c_{bc}|b\oplus c,c\rangle =$$

$$\downarrow \vdots$$

$$\frac{1}{\sqrt{2}}\sum_{b,c} c_{bc}(-1)^{(b+c)c}|b\oplus c,c\rangle$$

$$= \frac{1}{\sqrt{2}}\sum_{b,c} c_{b\oplus c,c}(-1)^{bc}|b,c\rangle$$

$$\downarrow M_Z$$

$$|a\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{ac}|c\rangle$$

$$\downarrow M_X$$

$$(-1)^{ab}|\bar{b}\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{ac}|c\rangle$$

omit this phase

Apply $P_a = \sum_d |\bar{d}, d\oplus a\rangle\langle\bar{d}, d\oplus a|$

$M_{X\otimes Z} \quad = \sum_d |\overline{d\oplus a}, d\rangle\langle\overline{d\oplus a}, d|$

$$\sum_d c_{d\oplus a,d}|\overline{d\oplus a},d\rangle$$

$$= \frac{1}{\sqrt{2}}\sum_{d,b} c_{d\oplus a,d}(-1)^{(d+a)b}|b,d\rangle$$

$$\downarrow M_Z$$

$$|b\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{(c+a)b}|c\rangle$$

$$\downarrow H\otimes I$$

$$|\bar{b}\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{(c+a)b}|c\rangle$$

$$\downarrow I\otimes Z^{a+b}$$

$$|\bar{b}\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{(c+a)b}(-1)^{c(a+b)}|c\rangle$$

$$= (-1)^{ab}|\bar{b}\rangle \otimes \sum_c c_{c\oplus a,c}(-1)^{ac}|c\rangle$$

omit this phase

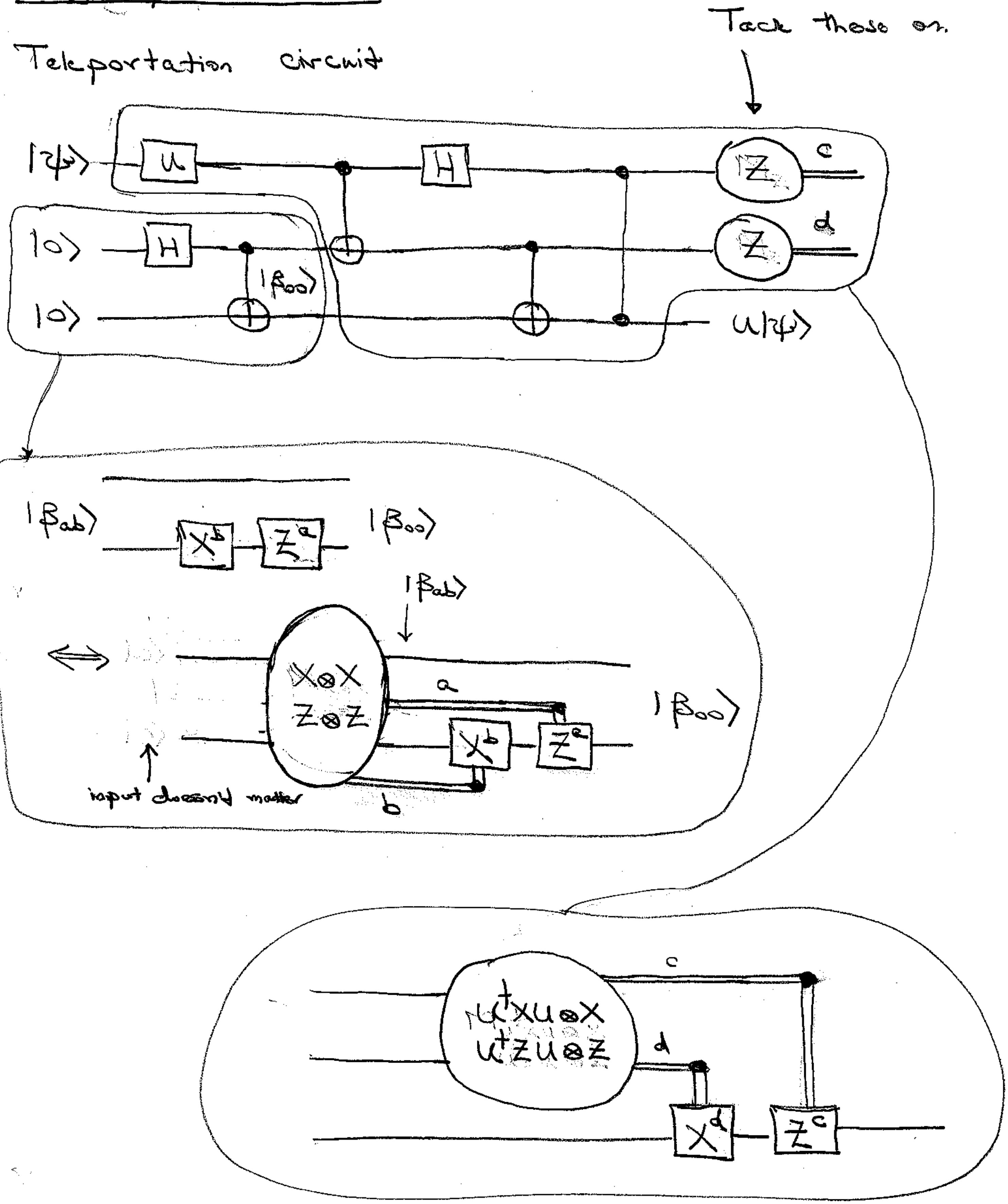
Same!

The phase $(-1)^{bc}$ is a crucial conditional phase flip.

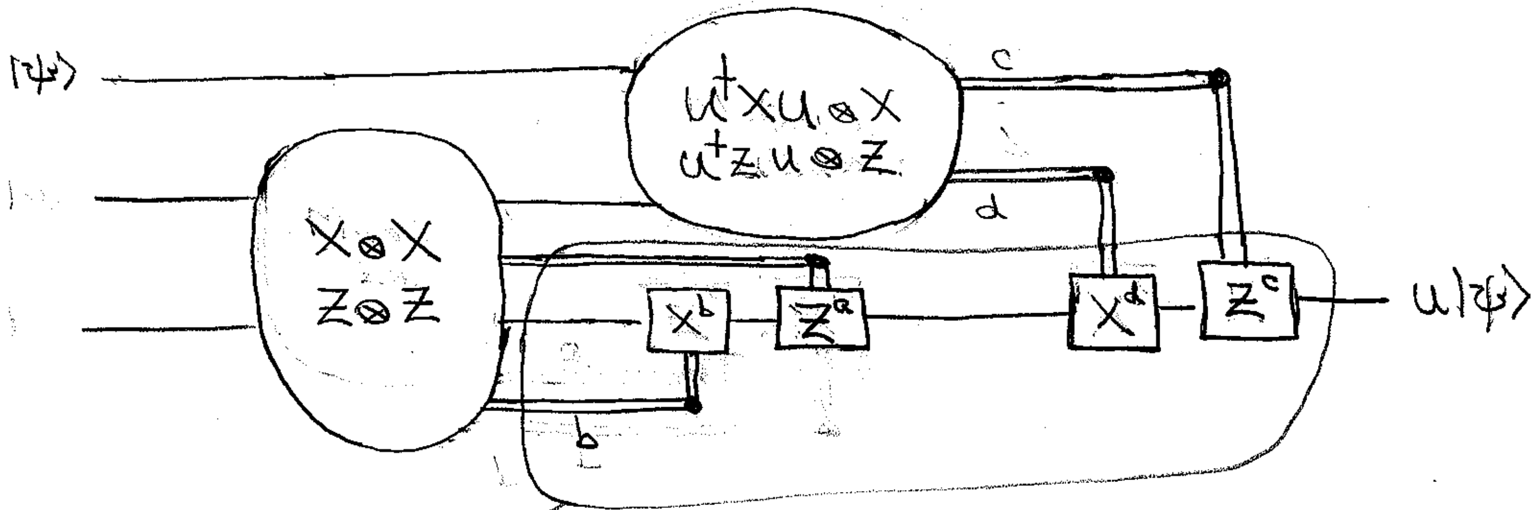
TQC

Single-qubit unitaries

Teleportation circuit



Single-qubit-unitary measurement circuit



But we really don't want to do these corrections, so what if we don't. Then the output state of the bottom qubit is

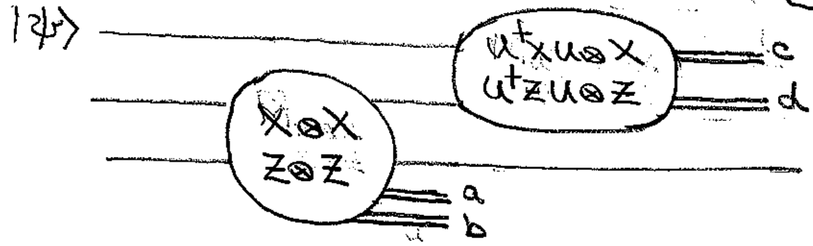
$$X^b Z^a X^d Z^c U|\psi\rangle = \underset{\substack{\uparrow \\ \text{meaningless} \\ \text{global phase} \\ \text{change}}}{(-1)^{ad}} \underbrace{X^{b+d} Z^{a+c}} U|\psi\rangle$$

4 possible Pauli errors. We don't correct them now, but rather keep track of them till the next gate involving this qubit, and remove them in the process of performing that gate. At the ultimate measurement, we take out the errors by measuring in a different Pauli basis.

If we do the classical controls, we have no error; if we don't, we have an error that is known.

Each working qubit has two classical bits attached to it, which specify the Pauli errors up to a sign. (11)

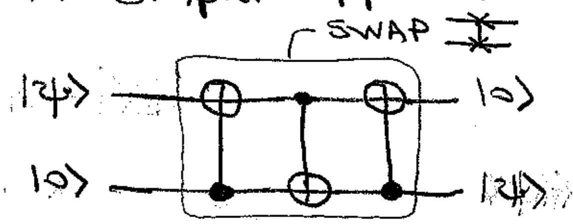
We end up with



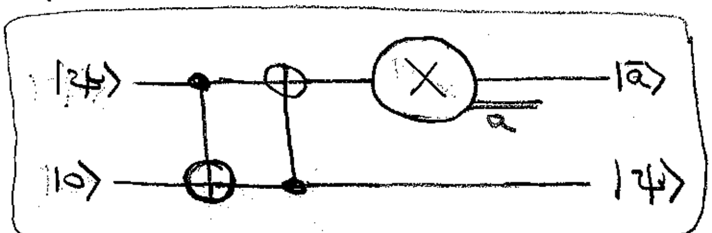
But we can do better if we don't start with teleportation as the primitive.

[Note that if U is a Pauli operator, we can simply incorporate it into the Pauli errors and not bother doing it. Moreover, if U conjugates Pauli errors to Pauli errors, i.e., is in the Clifford group (normalizer of the Pauli group), then we don't have to correct the errors; we can just map them forward to be corrected later.]

A simpler approach: use cheap teleportation circuits



$\Longrightarrow$

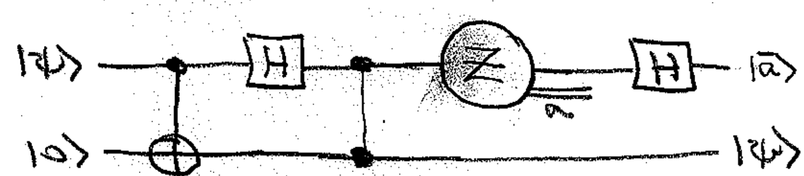


$|\psi\rangle \otimes |0\rangle = \sum_b c_b |b, 0\rangle \to \sum_b c_b |b, b\rangle = \frac{1}{\sqrt{2}} \sum_{a,b} c_b (-1)^{ab} |\bar a, b\rangle$

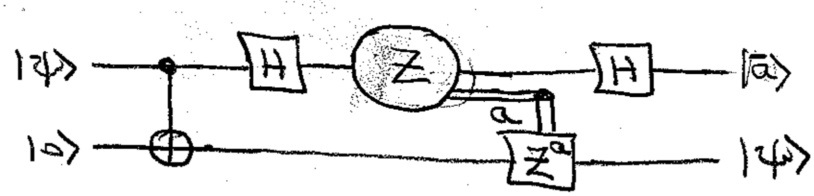
$= |\bar 0\rangle \otimes \frac{1}{\sqrt{2}}(c_0|0\rangle + c_1|1\rangle) + |\bar 1\rangle \otimes \frac{1}{\sqrt{2}}(c_0|0\rangle - c_1|1\rangle)$

$\to |\bar a\rangle \otimes \sum_b c_b (-1)^{ab} |b\rangle \to |\bar a\rangle \otimes \sum_b c_b |b\rangle = |\bar a\rangle \otimes |\psi\rangle$

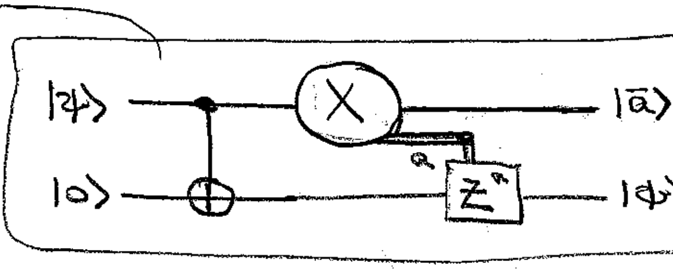
$\Longleftrightarrow$



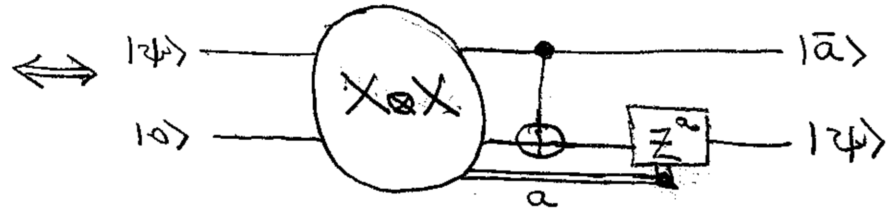
$\Longleftrightarrow$



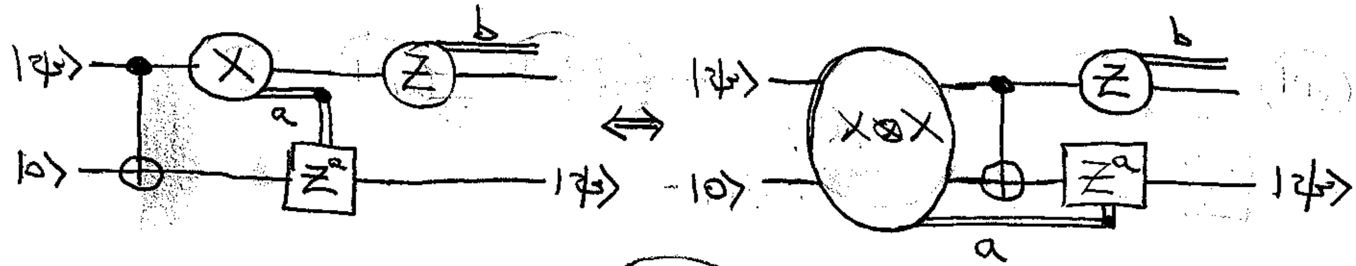
$\Longleftrightarrow$

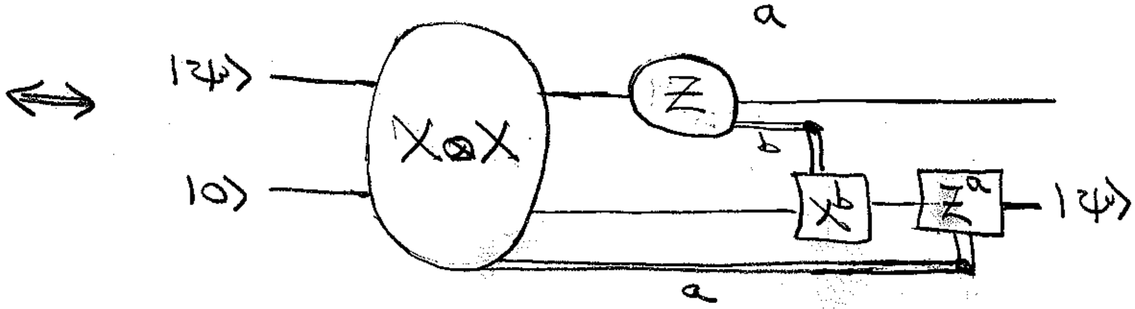


This is the primitive that looks like TP.

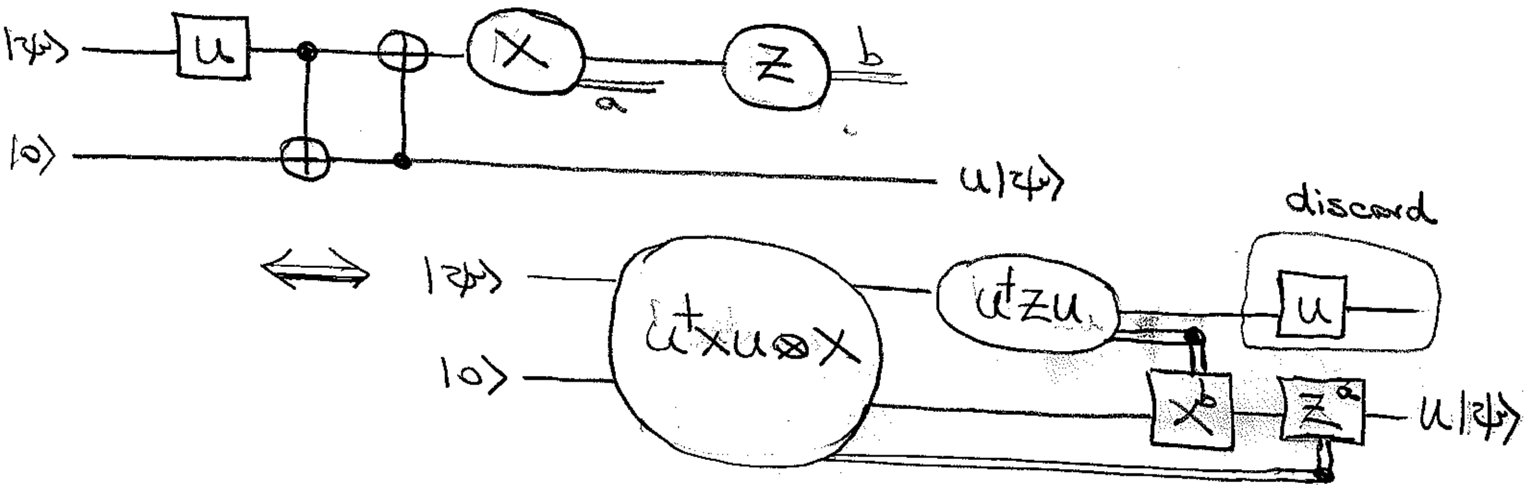


Need one more measurement to eat up the C-NOT.

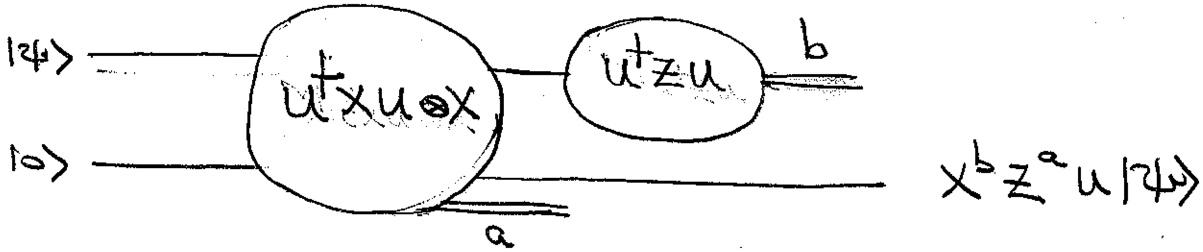




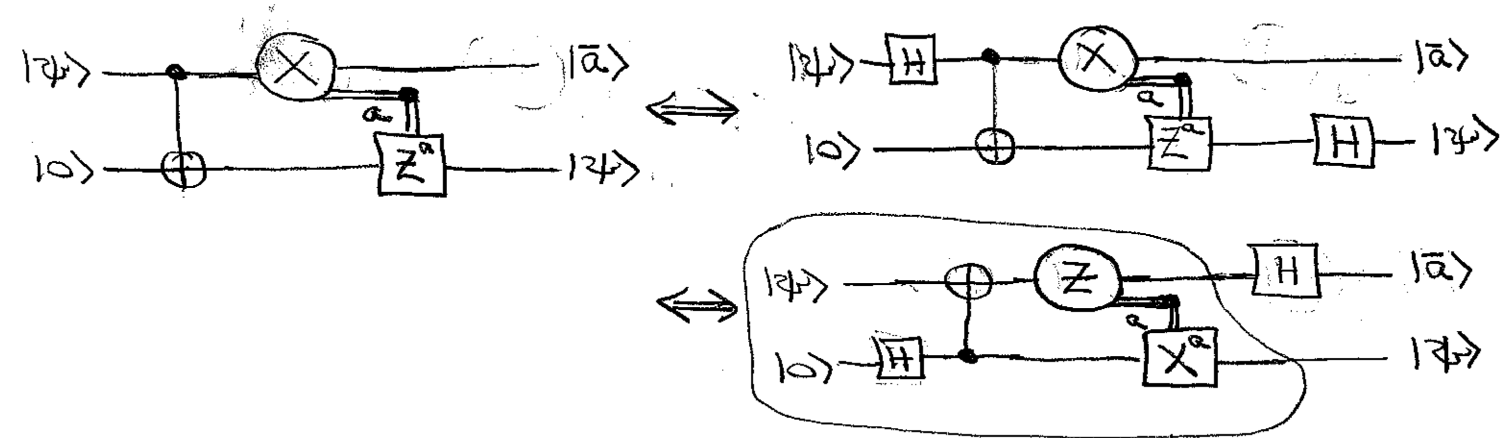
Putting in a unitary, we have



Leaving out the corrections, our single-qubit-unitary measurement circuit up to Pauli errors is

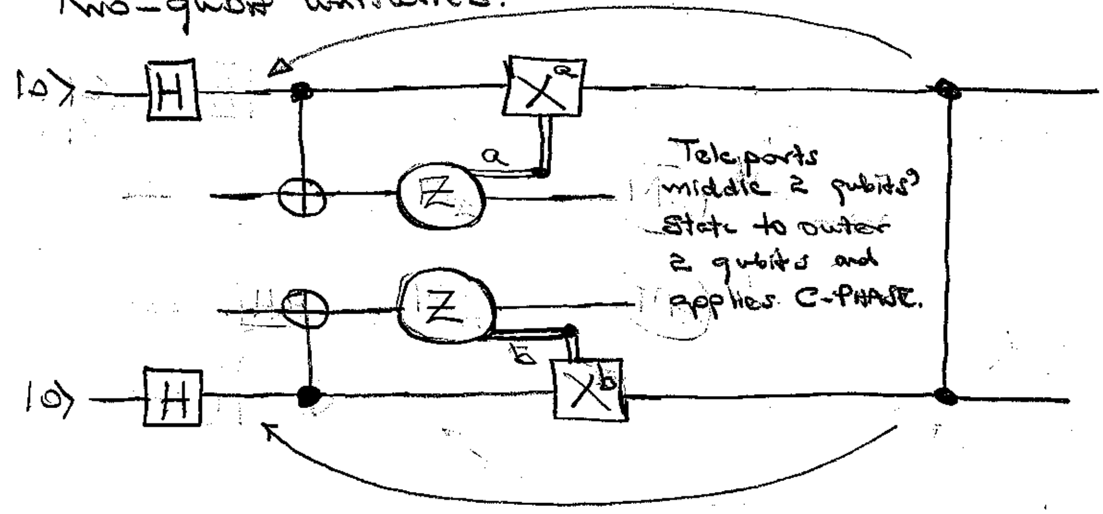


Notice that



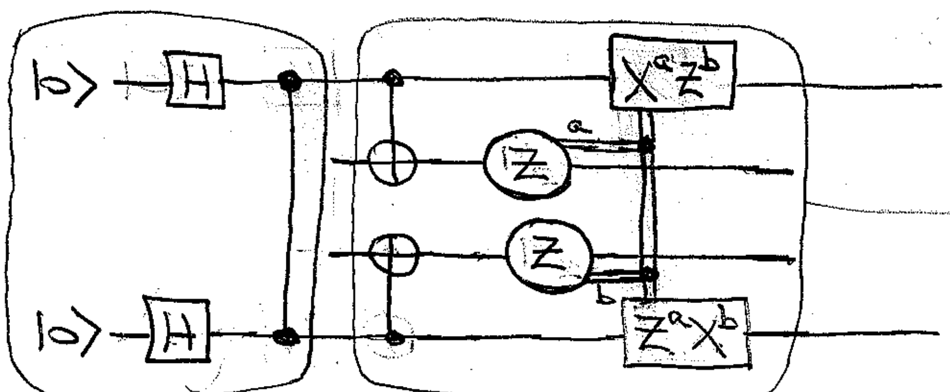
This is another TP primitive. It leads to the same measurement circuit.

Two-qubit unitaries: C-PHASE $\Lambda(Z)$



$$\Lambda(Z)\, X\otimes I\, \Lambda(Z) = X\otimes Z$$
$$\Lambda(Z)\, I\otimes X\, \Lambda(Z) = Z\otimes X$$
$$\Lambda(Z)\, Z\otimes I\, \Lambda(Z) = Z\otimes I$$
$$\Lambda(Z)\, I\otimes Z\, \Lambda(Z) = I\otimes Z$$

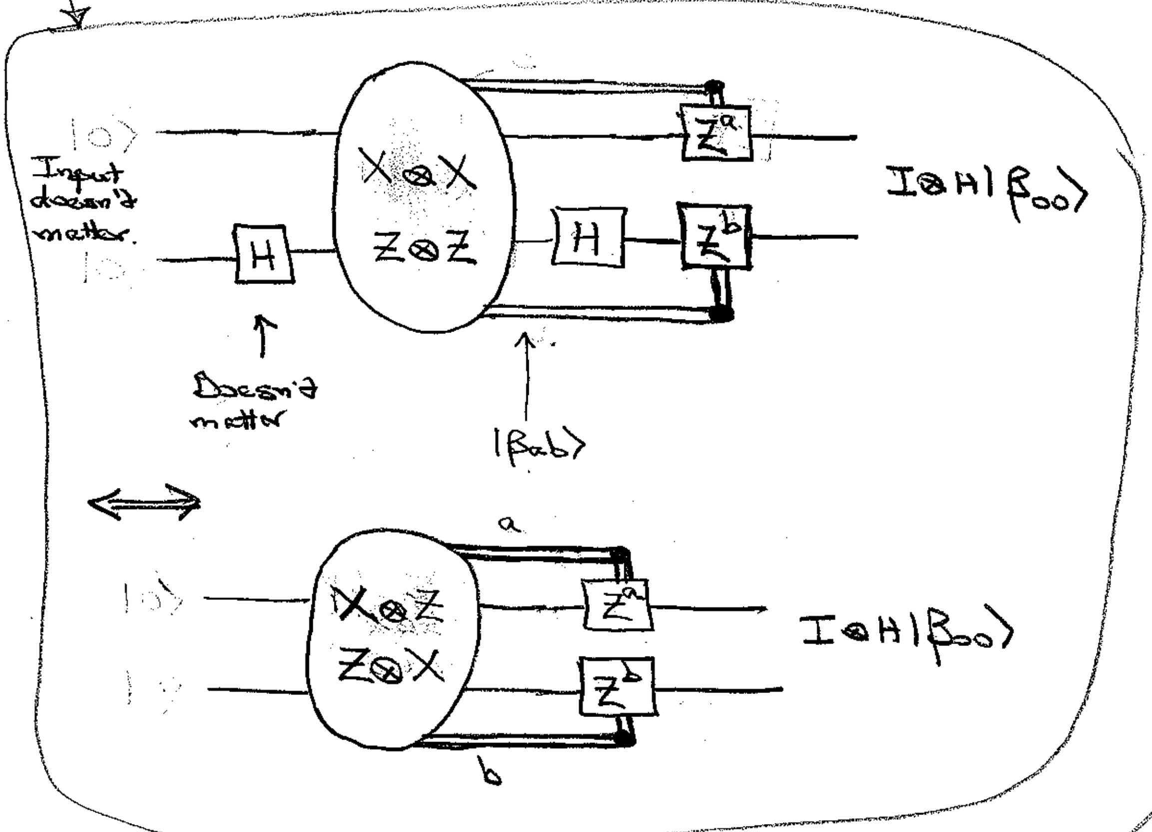
Controlled operations become $X^a Z^b$ on top and $Z^a X^b$ on bottom. (or opposite order for both).



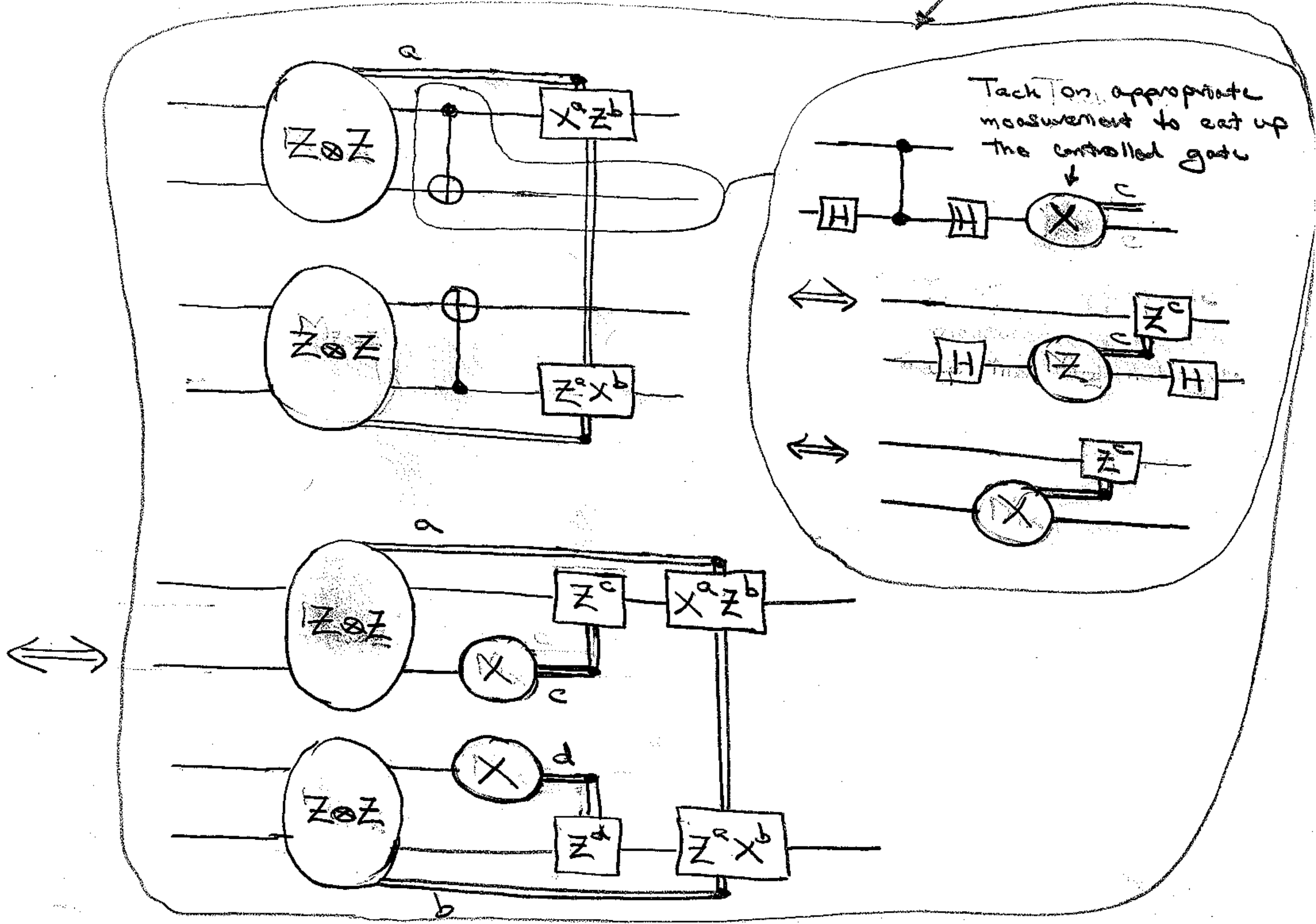
This is preparation of the entangled state

$$I\otimes H\,|\beta_{00}\rangle = (I\otimes H)(Z^a\otimes X^b)\,|\beta_{ab}\rangle = (Z^a\otimes Z^b)(I\otimes H)\,|\beta_{ab}\rangle$$

Do a measurement to prepare $(I\otimes H)|\beta_{ab}\rangle$, and then correct with $Z^a\otimes Z^b$

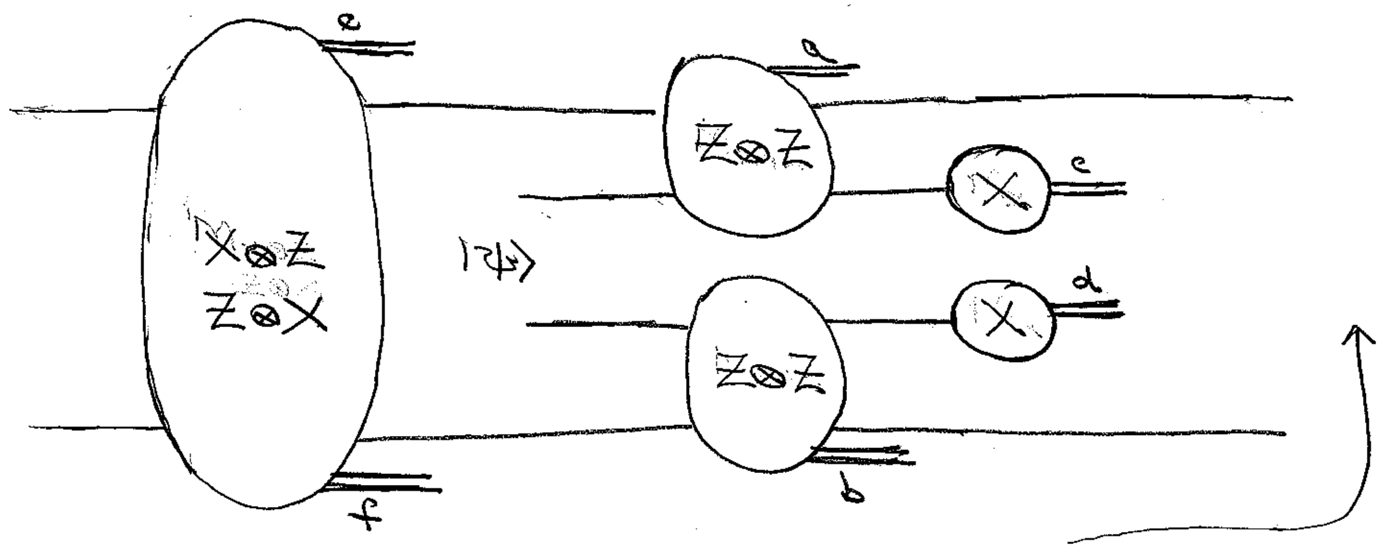


|0⟩
Input doesn't matter.
H
X⊗X
Z⊗Z
H
Z^a
Z^b
I⊗H|β00⟩
Doesn't matter
|βab⟩
a
|0⟩
X⊗Z
Z⊗X
Z^a
Z^b
b
I⊗H|β00⟩



a
Z⊗Z
X^a Z^b
Z⊗Z
Z^a X^b
Tack on appropriate measurement to eat up the controlled gate
H
H
X
c
Z^c
H
Z
H
Z^c
X
a
Z⊗Z
Z^c
X^a Z^b
X
c
X
d
Z⊗Z
Z^d
Z^a X^b
b

Putting it all together and leaving out the Pauli corrections, we have



Actually the input state has errors $X^r Z^s \otimes X^t Z^w$, and we need to correct these. The Z-errors change $c \to c+r$ and $d \to d+t$. The X-errors change $a \to a+s$ and $b \to b+w$. No change in measurement circuit.

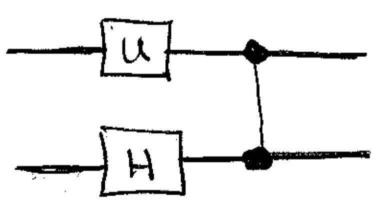
$$Z^{b+c+e} X^a \otimes Z^{f+d} X^b Z^a \Lambda(Z)|\psi\rangle$$

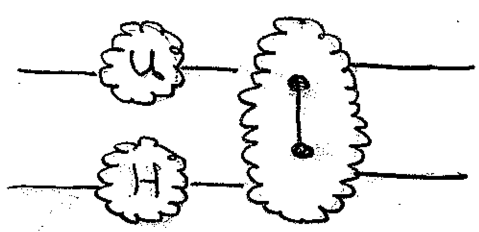
$$= (-1)^{ab} \underbrace{Z^{b+c+e} X^a \otimes Z^{a+d+f} X^b}_{\text{Pauli errors}} \Lambda(Z)|\psi\rangle$$

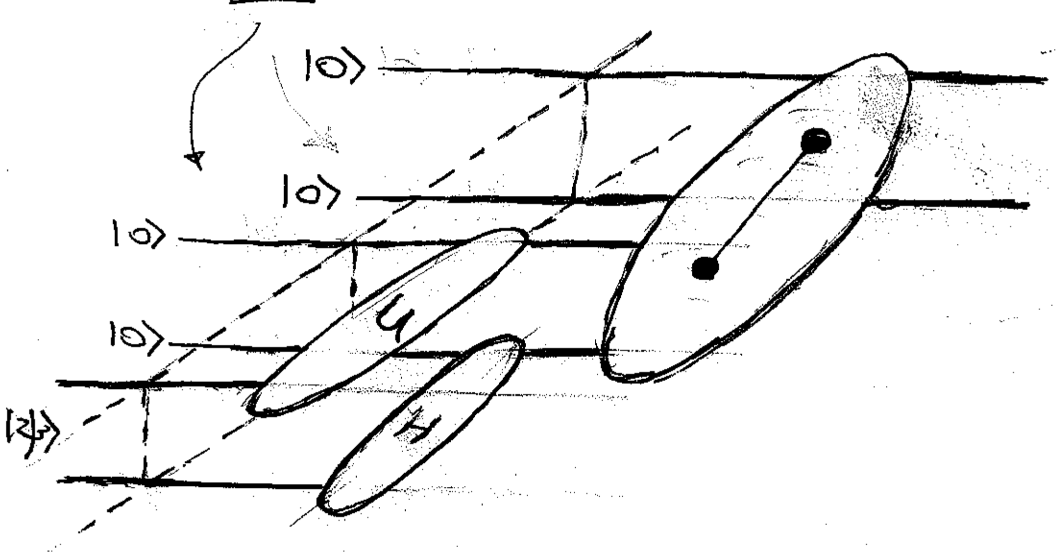
meaningless global phase

Clifford gate

Visualizing a measurement-based circuit







$\Lambda(Z)\, U \otimes H|\psi\rangle$

+ Pauli errors