2.3 **Qubit rotations.** An arbitrary unitary operator in a two-dimensional vector space can be written in the form

$$U = \exp\big(i\delta - i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2\big) = e^{i\delta}e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2}\ .$$

The phase $e^{i\delta}$ produces a global phase change, so we can discard it and write the general unitary operator as

$$U_R = e^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\theta/2}\ .$$

(a) What is the eigendecomposition of U_R?

(b) *Show* that

$$U_R = 1\cos(\theta/2) - i\mathbf{n}\cdot\boldsymbol{\sigma}\sin(\theta/2)\ .$$

(c) *Show* that

$$\begin{aligned} U_R^\dagger\boldsymbol{\sigma}U_R &= \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\sigma}) - \mathbf{n}\times(\mathbf{n}\times\boldsymbol{\sigma})\cos\theta + \mathbf{n}\times\boldsymbol{\sigma}\sin\theta \\ &= \boldsymbol{\sigma}\cos\theta + \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\sigma})(1-\cos\theta) + \mathbf{n}\times\boldsymbol{\sigma}\sin\theta \equiv R_{\mathbf{n}}(\theta)\boldsymbol{\sigma}\ . \end{aligned}$$

Here $R_{\mathbf{n}}(\theta)$ is the 3-dimensional orthogonal matrix that describes a rotation by angle θ about axis $\mathbf{n}$.

(d) Use the result of part (c) to *show* that U_R rotates any state $|\mathbf{m}\rangle$, i.e., that

$$U_R|\mathbf{m}\rangle = e^{i\phi(R,\mathbf{m})}|R\mathbf{m}\rangle\ ,$$

where $\phi(R,\mathbf{m})$ is a phase.

(e) *Show* that the unitary operator $\mathbf{n}\cdot\boldsymbol{\sigma}$ produces a 180° rotation about $\mathbf{n}$.

(f) The *Hadamard transform*,

$$H \equiv ie^{-i\mathbf{n}\cdot\boldsymbol{\sigma}\pi/2}\ ,$$

where $\mathbf{n} = (\mathbf{e}_x + \mathbf{e}_z)/\sqrt{2}$, rotates by 180° about the axis midway between the x and z axes. Show that the Hadamard transform has the matrix representation

$$H \leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\ .$$