Quantum information theory

Lectures 10-14

Quantum states: II - VI. Multiple systems and entanglement.

Multiple systems and the tensor product: (See additional notes)

Systems A (Hilbert space $\underset{\underset{D_A}{\uparrow}}{\mathcal{H}_A}$) and B (Hilbert space $\underset{\underset{D_B}{\uparrow}}{\mathcal{H}_B}$

Tensor-product Hilbert space $\underset{\underset{D_A D_B}{\uparrow}}{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{H}_B$

Orthonormal basis $|e_j\rangle \otimes |f_k\rangle = |e_j\rangle|f_k\rangle = |e_j, f_k\rangle$

$$\langle e_j, f_k | e_\ell, f_m\rangle = \langle e_j | e_\ell\rangle \langle f_k | f_m\rangle = \delta_{jk}\delta_{\ell m}$$

Vectors: $|\psi\rangle = \sum_{j,k} |e_j, f_k\rangle \underbrace{\langle e_j, f_k | \psi\rangle}_{c_{jk}}$

Operators: $A = \sum_{j,k,\ell,m} |e_j, f_k\rangle \underbrace{\langle e_j, f_k | A | e_\ell, f_m\rangle}_{A_{jk,\ell m}} \langle e_\ell, f_m|$

Reformulation of axioms of quantum mechanics.

1. Quantum states. Density operators ρ
2. Observables. $A = A^\dagger = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_j \lambda_j P_j = \sum_\lambda \lambda P_\lambda$

   $P(e_j) = \langle e_j|\rho|e_j\rangle = \mathrm{tr}(\rho P_j)$

   $P(\lambda) = \mathrm{tr}(\rho P_\lambda)$
   for degenerate observables

3. Post-measurement state.

   $$\frac{P_j \rho P_j}{\mathrm{tr}(\rho P_j)} = |e_j\rangle\langle e_j|$$

4. Dynamics: $i\hbar \frac{d\rho}{dt} = [H, \rho] \Leftrightarrow \rho(t) = e^{-iHt/\hbar} \rho(0) e^{iHt/\hbar}$
5. Hilbert space for multiple systems is the tensor-product space.

Gleason's theorem. For $D \geq 3$, if a function f from 1-d' projectors to $[0,1]$ satisfies $\sum_j f(P_j) = 1$, whenever $\sum_j P_j = 1$, then there exists a density operator ρ such that $f(P) = \mathrm{tr}(\rho P)$.

Axiomatization of qm. State space from Hilbert-space structure of quantum measurements.

Why Hilbert space?

Why noncontextual probabilities?

Partial trace and marginal density operators (see additional notes)

Operator
$$O = \sum_{j,k,l,m} |e_j, f_k\rangle \underbrace{\langle e_j, f_k | O | e_l, f_m \rangle}_{O_{jk,lm}} \langle e_l, f_m|$$

Partial trace:

$$\mathrm{tr}_B(O) = \sum_k \underbrace{\langle f_k | O | f_k \rangle}_{\text{operator on A}} = \sum_{j,l} |e_j\rangle\langle e_l| \sum_k O_{jk,lk}$$

Measure in basis $|e_j\rangle$ on A:

$$P(e_j) = \sum_k P(e_j, f_k)$$
$$= \sum_k \langle e_j, f_k | \rho | e_j, f_k \rangle = \langle e_j | \underbrace{\sum_k \langle f_k | \rho | f_k \rangle}_{} | e_j \rangle$$

$$= \mathrm{tr}_B(\rho) \equiv \rho_A = \begin{pmatrix} \text{marginal or} \\ \text{reduced density} \\ \text{operator of A} \end{pmatrix}$$

$$= \mathrm{tr}\Big(\rho\, \underbrace{|e_j\rangle\langle e_j|}_{P_j} \otimes \underbrace{\sum_k |f_k\rangle\langle f_k|}_{1_B}\Big) = \mathrm{tr}(\rho\, \underbrace{P_j \otimes 1_B})$$

projection operator for a measurement on A

$$\Big\downarrow$$

$$= \mathrm{tr}(\rho_A P_j)$$

Two qubits

Orthonormal (standard) basis:

$$\left.\begin{aligned} |00\rangle &= |0\rangle \otimes |0\rangle \\ |01\rangle &= |0\rangle \otimes |1\rangle \\ |10\rangle &= |1\rangle \otimes |0\rangle \\ |11\rangle &= |1\rangle \otimes |1\rangle \end{aligned}\right\} \quad |x\rangle = |x_1\rangle \otimes |x_2\rangle$$

Bell (orthonormal) basis:

$$\left\{\begin{aligned} |\Phi^\pm\rangle &= \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle) = \begin{matrix} |\beta_{00}\rangle \\ |\beta_{10}\rangle \end{matrix} \\ |\Psi^\pm\rangle &= \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle) = \begin{matrix} |\beta_{01}\rangle \\ |\beta_{11}\rangle \end{matrix} \end{aligned}\right.$$

Singlet vs. triplet

$Z \otimes Z$, $X \otimes X$, $Y \otimes Y$ Commute

$Z \otimes Z |\Phi^\pm\rangle = |\Phi^\pm\rangle$ "parity" bit:
$Z \otimes Z |\Psi^\pm\rangle = -|\Psi^\pm\rangle$ Φ vs. Ψ

$X \otimes X |\Phi^\pm\rangle = \pm|\Phi^\pm\rangle$ "phase" bit:
$X \otimes X |\Psi^\pm\rangle = \pm|\Psi^\pm\rangle$ + vs. −

$$|\beta_{ab}\rangle = \frac{1}{\sqrt{2}}\left(|0b\rangle + (-1)^a |1\bar{b}\rangle\right)$$

phase bit → a, parity bit → b; $\bar{b}$ ← $1 \oplus b$

$Z \otimes Z |\beta_{ab}\rangle = (-1)^b |\beta_{ab}$

$X \otimes X |\beta_{ab}\rangle = (-1)^a |\beta_{ab}$

Marginal density operators: $\rho_A = \rho_B = \frac{1}{2}1$

Entangled: $|\Psi\rangle \neq \underbrace{|\psi_A\rangle \otimes |\psi_B\rangle}_{\text{product pure state}}$

Superdense coding:

$$Z\otimes 1\,|\beta_{00}\rangle = |\beta_{10}\rangle = 1\otimes Z\,|\beta_{00}\rangle$$

$$X\otimes 1\,|\beta_{00}\rangle = |\beta_{01}\rangle = 1\otimes X\,|\beta_{00}\rangle \Leftrightarrow \boxed{\begin{aligned}|\beta_{ab}\rangle &= Z^a X^b\otimes 1\,|\beta_{00}\rangle\\ &= 1\otimes X^b Z^a\,|\beta_{00}\rangle\end{aligned}}$$

$$\underbrace{ZX}_{iY}\otimes 1\,|\beta_{00}\rangle = |\beta_{11}\rangle = 1\otimes \underbrace{XZ}_{-iY}\,|\beta_{00}\rangle$$

Alice and Bob each have one qubit of an entangled pair in state $|\beta_{00}\rangle$. Alice can encode two bits on joint state using 1, Z, X, or ZX on her qubit. Alice sends her qubit to Bob, who reads out the two bits using a measurement in the Bell basis.

Magnetic-field-in-a-box version

Pauli representation: Any $\rho = \frac{1}{4}\sum_{\alpha\beta} \rho_{\alpha\beta}\,\sigma_\alpha\otimes\sigma_\beta$,

$$\rho_{\alpha\beta} = \mathrm{tr}(\rho\,\sigma_\alpha\otimes\sigma_\beta), \quad \rho_{00} = 1$$

$$|\Phi^+\rangle\langle\Phi^+| = \frac{1}{2}\Big(\underbrace{|00\rangle\langle 00|}_{\frac{1}{2}(1+Z)\otimes\frac{1}{2}(1+Z)} + \underbrace{|11\rangle\langle 11|}_{\frac{1}{2}(1-Z)\otimes\frac{1}{2}(1-Z)} + \underbrace{|00\rangle\langle 11|}_{\frac{1}{2}(X+iY)\otimes\frac{1}{2}(X+iY)} + \underbrace{|11\rangle\langle 00|}_{\frac{1}{2}(X-iY)\otimes\frac{1}{2}(X-iY)}\Big)$$

$$= \frac{1}{4}(1\otimes 1 + Z\otimes Z + X\otimes X - Y\otimes Y)$$

$$|\beta_{00}\rangle\langle\beta_{00}| = |\Phi^+\rangle\langle\Phi^+| = \frac{1}{4}(1\otimes 1 + Z\otimes Z + X\otimes X - Y\otimes Y)$$

$$|\beta_{10}\rangle\langle\beta_{10}| = |\Phi^-\rangle\langle\Phi^-| = \frac{1}{4}(1\otimes 1 + Z\otimes Z - X\otimes X + Y\otimes Y)$$

$$|\beta_{01}\rangle\langle\beta_{01}| = |\Psi^+\rangle\langle\Psi^+| = \frac{1}{4}(1\otimes 1 - Z\otimes Z + X\otimes X + Y\otimes Y)$$

$$|\beta_{11}\rangle\langle\beta_{11}| = |\Psi^-\rangle\langle\Psi^-| = \frac{1}{4}(1\otimes 1 - Z\otimes Z - X\otimes X - Y\otimes Y) \leftarrow \text{rotationally invariant}$$

Z is a $180°$ rotation about z, so $ZXZ = -X$, $ZYZ = -Y$ and $ZZZ = Z$. Likewise, $XXX = X$, $XYX = -Y$ and $XZX = -Z$

$$U\otimes U\,\sigma_j\otimes\sigma_j\,U^\dagger\otimes U^\dagger = \sigma_k R_{kj}\otimes\sigma_l R_{lj} = \sigma_k\otimes\sigma_l\,(RR^T)_{kl} = \sigma_k\otimes\sigma_k$$

Generally, we can write the Pauli representations as

$$|\beta_{ab}\rangle\langle\beta_{ab}| = \frac{1}{4}\left(1\otimes 1 + (-1)^{b}\, Z\otimes Z + (-1)^{a}\, X\otimes X - (-1)^{a+b}\, Y\otimes Y\right)$$

These are the eigenvalues below

Inverting these relations, we find

$$1\otimes 1 = +|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{01}\rangle\langle\beta_{01}| + |\beta_{11}\rangle\langle\beta_{11}|$$

$$Z\otimes Z = +|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{10}\rangle\langle\beta_{10}| - |\beta_{01}\rangle\langle\beta_{01}| - |\beta_{11}\rangle\langle\beta_{11}|$$

$$X\otimes X = +|\beta_{00}\rangle\langle\beta_{00}| - |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{01}\rangle\langle\beta_{01}| - |\beta_{11}\rangle\langle\beta_{11}|$$

$$Y\otimes Y = \underbrace{-|\beta_{00}\rangle\langle\beta_{00}| + |\beta_{10}\rangle\langle\beta_{10}| + |\beta_{01}\rangle\langle\beta_{01}| - |\beta_{11}\rangle\langle\beta_{11}|}$$

These are commuting operators
Any two of X⊗X, Y⊗Y, and Z⊗Z is a complete set of commuting operators

These are the simultaneous eigendecompositions in terms of Bell states.

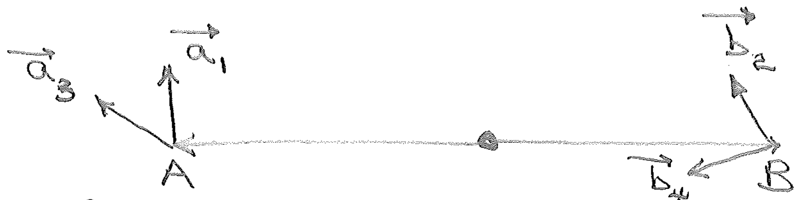
$Z\otimes Z|\beta_{ab}\rangle = (-1)^{b}|\beta_{ab}\rangle$; a measurement of $Z\otimes Z$ determines the parity bit.

$X\otimes X|\beta_{ab}\rangle = (-1)^{a}|\beta_{ab}\rangle$; a measurement of $X\otimes X$ determines the phase bit.

$$Y\otimes Y|\beta_{ab}\rangle = -(-1)^{a+b}|\beta_{ab}\rangle$$

$$= -ZX\otimes ZX = -XZ\otimes XZ$$

CHSH Bell inequality:



Realism: Values of $\sigma_{\vec{a}}, \sigma_{\vec{b}} = \pm 1$ are objective properties of A and B.

No-disturbance: Measurement on A does not disturb objective values of B

$\uparrow$

enforced by locality

Local realism

$$C(\vec{a},\vec{b}) = \langle \sigma_j a_j \otimes \sigma_k b_k \rangle = a_j b_k \underbrace{\langle \sigma_j \otimes \sigma_k \rangle}_{-\delta_{jk} \text{ for singlet}} = -\vec{a}\cdot\vec{b}$$

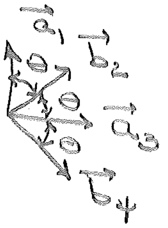
$$\left| \sigma_{\vec{a}_1}(\sigma_{\vec{b}_2} - \sigma_{\vec{b}_4}) + \sigma_{\vec{a}_3}(\sigma_{\vec{b}_2} + \sigma_{\vec{b}_4}) \right| = 2$$

$$|\langle \quad \rangle| \leq \langle | \quad | \rangle = 2$$

$$\boxed{\left| \overbrace{\underbrace{C(\vec{a}_1,\vec{b}_2)}_{\langle \sigma_{\vec{a}_1} \sigma_{\vec{b}_2} \rangle} + C(\vec{a}_3,\vec{b}_2) + C(\vec{a}_3,\vec{b}_4) - C(\vec{a}_1,\vec{b}_4)}^{=: S} \right| \leq 2}$$

CHSH Bell inequality

QM:



$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$ Singlet: rotationally invariant

$$C(\vec{a},\vec{b}) = \langle \vec{\sigma}\cdot\vec{a} \otimes \vec{\sigma}\cdot\vec{b} \rangle = -\vec{a}\cdot\vec{b} = -\cos\theta_{ab}$$

$$\Rightarrow \underline{S = |3\cos\theta - \cos 3\theta| \longrightarrow 2\sqrt{2}} \text{ for } \theta = 45°$$

For $\theta = 45°$, $\vec{a}_1 = \vec{e}_x$, $\vec{b}_2 = \frac{1}{\sqrt{2}}(\vec{e}_y + \vec{e}_x)$, $\vec{a}_3 = \vec{e}_y$, $\vec{b}_4 = \frac{1}{\sqrt{2}}(\vec{e}_y - \vec{e}_x)$

$$S = \langle X \otimes \tfrac{1}{\sqrt{2}}(Y+X) + Y \otimes \tfrac{1}{\sqrt{2}}(Y+X) + Y \otimes \tfrac{1}{\sqrt{2}}(Y-X) - X \otimes \tfrac{1}{\sqrt{2}}(Y-X) \rangle$$

$$= \sqrt{2}\langle X \otimes X + Y \otimes Y \rangle$$

$$= 2\sqrt{2}\langle |\beta_{01}\rangle\langle\beta_{01}| - |\beta_{11}\rangle\langle\beta_{11}| \rangle$$

Maximal violation for $|\beta_{01}\rangle$ (correlations) or $|\beta_{11}\rangle$ (anticorrelations)

Teleportation: Alice and Bob each has one of a pair of qubits in entangled state $|\beta_{00}^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Victor hands to Alice a qubit in state $|\psi^V\rangle$.

Total state: $|\Psi\rangle = |\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle$

$$= \sum_{a,b} |\beta_{ab}^{VA}\rangle \underbrace{\langle \beta_{ab}^{VA} | \Psi \rangle}_{\text{vector in Bob's Hilbert space}}$$

magnitude times "relative state"

$$\langle \beta_{00}^{VA} | \Psi \rangle = \langle \beta_{00}^{VA} | \left(|\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle \right)$$

$$= \frac{1}{2}\left(\langle 00| + \langle 11| \right)\left(|\psi^V\rangle \otimes (|00\rangle + |11\rangle) \right)$$

$$= \frac{1}{2}\left(\langle 0|\psi^V\rangle |0\rangle + \langle 1|\psi^V\rangle |1\rangle \right)$$

relative state in B

$$= \frac{1}{2}|\psi^B\rangle$$

$|\psi\rangle$ belonging to B

$$\langle \beta_{ab}^{VA} | \Psi \rangle = \left(\langle \beta_{00}^{VA} | X^b Z^a \otimes 1 \left(|\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle \right) \right)$$

$$= \langle \beta_{00}^{VA} | \left(X^b Z^a |\psi^V\rangle \otimes |\beta_{00}^{AB}\rangle \right)$$

compare

$$= \frac{1}{2} X^b Z^a |\psi^B\rangle \quad \text{relative state in B}$$

$$|\Psi\rangle = \frac{1}{2}\sum_{a,b} |\beta_{ab}^{VA}\rangle \otimes X^b Z^a |\psi^B\rangle$$

$$= \frac{1}{2}\Big(|\beta_{00}^{VA}\rangle \otimes |\psi^B\rangle + |\beta_{10}^{VA}\rangle \otimes Z|\psi^B\rangle + |\beta_{01}^{VA}\rangle \otimes X|\psi^B\rangle$$
$$+ |\beta_{11}^{VA}\rangle \otimes XZ|\psi^B\rangle \Big)$$

Alice measures in the Bell basis (all 4 results equally likely; no trace of $|\psi^V\rangle$ left in VA), sends 2 bits, a and b, to Bob, who applies the appropriate unitary, $Z^a X^b$, to leave his qubit in state $|\psi^V\rangle$.

Nonsquare matrices: add rows and columns of zeroes to make matrix square.

Polar decomposition and singular values. Given any operator A, $A^\dagger A$ and $AA^\dagger$ are positive operators with the same (nonnegative) eigenvalues, whose square roots are the singular values of A. Moreover, there exists a unitary operator U such that

$$A = U\sqrt{A^\dagger A} = \sqrt{AA^\dagger}\,U. \longleftarrow \text{polar decomposition}$$

U is unique if $A^\dagger A$ is invertible. The eigenvectors $|e_j\rangle$ of $A^\dagger A$ are called the right singular vectors of A, and the eigenvectors $|f_j\rangle$ of $AA^\dagger$ are called the left singular vectors of A.

$\boxed{A \text{ is normal} \Leftrightarrow [U, A^\dagger A] = 0.}$

Proof:

Eigendecomposition $A^\dagger A = \sum_j \lambda_j^2 |e_j\rangle\langle e_j|$, $A^\dagger A|e_j\rangle = \lambda_j^2|e_j\rangle$.

Let $S_{A^\dagger A}$ be the support of $A^\dagger A$, and let $N_{A^\dagger A}$ be the null subspace. Now

$$AA^\dagger(A|e_j\rangle) = A(A^\dagger A|e_j\rangle) = \lambda_j^2 A|e_j\rangle$$

$\Rightarrow A|e_j\rangle$ is an eigenvector of $AA^\dagger$ with eigenvalue λ_j^2.

Two cases:

(i) $\lambda_j = 0$ ($|e_j\rangle \in N_{A^\dagger A}$): $\langle e_j|A^\dagger A|e_j\rangle = 0 \Rightarrow A|e_j\rangle = 0$

(ii) $\lambda_j \neq 0$ ($|e_j\rangle \in S_{A^\dagger A}$): Define $|f_j\rangle \equiv \dfrac{A|e_j\rangle}{\langle e_j|A^\dagger A|e_j\rangle^{1/2}} = A|e_j\rangle/\lambda_j$.

Notice that $\langle f_j|f_k\rangle = \langle e_j|A^\dagger A|e_k\rangle/\lambda_j\lambda_k = \delta_{jk}$, so the vectors $|f_j\rangle$ are orthonormal eigenvectors of $AA^\dagger$.

Now

$$(AA^\dagger)^2 = A(A^\dagger A)A^\dagger = \sum_j \lambda_j^2 A|e_j\rangle\langle e_j|A^\dagger$$

$$= \sum_{\lambda_j \neq 0} \lambda_j^2 \underbrace{A|e_j\rangle\langle e_j|A^\dagger}_{\lambda_j^2 |f_j\rangle\langle f_j|}$$

$$= \underbrace{\sum_{\lambda_j \neq 0} \lambda_j^4 |f_j\rangle\langle f_j|}_{\text{eigendecomposition of } (AA^\dagger)^2}$$

$$\Longrightarrow \quad AA^\dagger = \underbrace{\sum_{\lambda_j \neq 0} \lambda_j^2 |f_j\rangle\langle f_j|}_{\text{eigendecomposition of } AA^\dagger} \quad \Longrightarrow$$ The vectors $|f_j\rangle$ span the support $S_{AA^\dagger}$.

Now complete the orthonormal basis $|f_j\rangle$ by adding on any orthonormal vectors in the null subspace of $AA^\dagger$. For all j, we can write

$$A|e_j\rangle = \lambda_j |f_j\rangle = \begin{cases} \lambda_j |f_j\rangle, & \lambda_j \neq 0 \\ 0, & \lambda_j = 0 \end{cases}.$$

Thus A has the form

$$A = \sum_{j,k} |f_j\rangle \underbrace{\langle f_j| A|e_k\rangle}_{\lambda_j \langle f_j|f_k\rangle = \lambda_j \delta_{jk}} \langle e_k| = \sum_j \lambda_j \underset{\substack{\uparrow \\ \text{left singular vectors}}}{|f_j\rangle} \underset{\substack{\uparrow \\ \text{right singular vectors}}}{\langle e_j|}$$

Summary till now:

$$A = \sum_j \lambda_j |f_j\rangle\langle e_j| \qquad A^\dagger A = \sum_j \lambda_j^2 |e_j\rangle\langle e_j|$$

$$A^\dagger = \sum_j \lambda_j |e_j\rangle\langle f_j| \qquad AA^\dagger = \sum_j \lambda_j^2 |f_j\rangle\langle f_j|$$

$$\begin{array}{ll} S_{A^\dagger A}\left\{\begin{array}{l} |e_1\rangle \\ \vdots \\ |e_r\rangle \end{array}\right. & \begin{array}{l} \lambda_1|f_1\rangle \\ \vdots \\ \lambda_r|f_r\rangle \end{array} \\ & \xrightarrow{\quad A \quad} \\ N_{A^\dagger A}\left\{\begin{array}{l} |e_{r+1}\rangle \\ \vdots \\ |e_D\rangle \end{array}\right. & \begin{array}{l} 0 \\ \vdots \\ 0 \end{array} \end{array} \qquad \begin{array}{ll} S_{AA^\dagger}\left\{\begin{array}{l} |f_1\rangle \\ \vdots \\ |f_r\rangle \end{array}\right. & \begin{array}{l} \lambda_1|e_1\rangle \\ \vdots \\ \lambda_r|e_r\rangle \end{array} \\ & \xrightarrow{\quad A^\dagger \quad} \\ N_{AA^\dagger}\left\{\begin{array}{l} |f_{r+1}\rangle \\ \vdots \\ |f_D\rangle \end{array}\right. & \begin{array}{l} 0 \\ \vdots \\ 0 \end{array} \end{array}$$

r is the <u>rank</u> of A

Finishing off, we can define the unitary operator $U = \sum_j |f_j\rangle\langle e_j|$, i.e., $U|e_j\rangle = |f_j\rangle$ (unitary because U maps orthonormal basis $|e_j\rangle$ to orthonormal basis $|f_j\rangle$). Thus

$$A = \sum_j \lambda_j |f_j\rangle\langle e_j|$$

$$= \Big(\sum_j |f_j\rangle\langle e_j|\Big)\sqrt{A^\dagger A} = U\sqrt{A^\dagger A}$$

$$= \sqrt{AA^\dagger}\Big(\sum_j |f_j\rangle\langle e_j|\Big) = \sqrt{AA^\dagger}\,U$$

The freedom in U is the freedom in how it maps $N_{A^\dagger A}$ to $N_{AA^\dagger}$. There is no freedom if $A^\dagger A$ has no zero eigenvalues, for then $A^\dagger A$ is invertible and

$$U = A\underbrace{(A^\dagger A)^{-1/2}}_{(\sqrt{A^\dagger A})^{-1}} .$$

Going directly from $A = U\sqrt{A^\dagger A}$ to $A = \sqrt{AA^\dagger}\,U$:

$$A = U\sqrt{A^\dagger A} \Rightarrow AA^\dagger = U\sqrt{A^\dagger A}\sqrt{A^\dagger A}\,U^\dagger = UA^\dagger A U^\dagger$$

$$\Rightarrow \sqrt{AA^\dagger} = U\sqrt{A^\dagger A}\,U^\dagger$$

Singular-value decomposition. Given an operator A and an arbitrary orthonormal basis $|g_j\rangle$, there exist unitary operators V and W such that $A = V\Lambda W^\dagger$, where

$$\Lambda = \sum_j \lambda_j |g_j\rangle\langle g_j|$$

$$\boxed{A = \sum_j \lambda_j \underbrace{V|g_j\rangle}_{|f_j\rangle} \underbrace{\langle g_j|W^\dagger}_{\langle e_j|}}$$

is diagonal in the basis $|g_j\rangle$, its eigenvalues being the singular values of A.

Proof:

$$\underset{\substack{\uparrow \\ \text{polar} \\ \text{decomposition}}}{A} = U\sqrt{A^\dagger A} = \underbrace{U\sum_j \lambda_j |e_j\rangle\langle e_j|}_{\substack{\text{eigendecomposition} \\ \text{of } \sqrt{A^\dagger A}}} = \overbrace{UW}^{\equiv V}\overbrace{\left(\sum_j \lambda_j |g_j\rangle\langle g_j|\right)}^{\equiv \Lambda}W^\dagger$$

Unitary W defined by $W|g_j\rangle = |e_j\rangle$

Notice that $V|g_j\rangle = |f_j\rangle$

Relative states. $|\phi_\alpha\rangle = \sqrt{p_\alpha}\,|\psi_\alpha\rangle$ is a vector in $\mathcal{H}_A$

Bipartite pure state

$$|\Psi\rangle = \sum_\alpha \underset{\substack{\uparrow \\ \text{orthonormal basis for } B}}{|f_\alpha\rangle} \overbrace{\langle f_\alpha|\Psi\rangle} = \sum_\alpha \sqrt{p_\alpha}\underset{\substack{\uparrow \\ \text{relative state}}}{|\psi_\alpha\rangle} \otimes |f_\alpha\rangle$$

$$\rho_A = \mathrm{tr}_B\big(\underbrace{|\Psi\rangle\langle\Psi|}_{\sum_{j,k}\sqrt{p_\alpha p_\beta}\,|\psi_\alpha\rangle\langle\psi_\beta|\otimes|f_\alpha\rangle\langle f_\beta|}\big) = \sum_{\alpha,\beta}\sqrt{p_\alpha p_\beta}\,|\psi_\alpha\rangle\langle\psi_\beta|\, \underbrace{\mathrm{tr}\big(|f_\alpha\rangle\langle f_\beta|\big)}_{\langle f_\beta|f_\alpha\rangle = \delta_{\alpha\beta}}$$

$$\text{So } \rho_A = \sum_\alpha p_\alpha \underset{\substack{\uparrow \\ \text{not necessarily} \\ \text{orthonormal}}}{|\psi_\alpha\rangle\langle\psi_\alpha|} \longleftarrow \text{ensemble decomposition of } \rho_A$$

Schmidt decomposition. Any bipartite pure state can be written as

special relative-state decomposition

$$|\Psi\rangle = \sum_j \sqrt{\lambda_j}\, |e_j\rangle \otimes |f_j\rangle .$$

orthonormal vectors in $\mathcal{H}_B$

orthonormal vectors in $\mathcal{H}_A$

nonnegative Schmidt coefficients

Schmidt vectors

As a consequence, $\rho_A = \sum_j \lambda_j |e_j\rangle\langle e_j|$ and $\rho_B = \sum_j \lambda_j |f_j\rangle\langle f_j|$ have the same eigenvalues.

What's the point? Schmidt coefficients provide a complete characterization of bipartite pure-state entanglement. Schmidt coefficients are invariant under local unitaries. Schmidt vectors can be mapped to any pair of bases by local unitaries. There is no Schmidt-like decomposition for 3 or more systems.

Proof: Diagonalize $\rho_B = \mathrm{tr}_A(|\Psi\rangle\langle\Psi|) = \sum_j \lambda_j |f_j\rangle\langle f_j|$. Form the relative-state decomposition wrt to the vectors $|f_j\rangle$, $|\Psi\rangle = \sum_j \underbrace{|\phi_j\rangle}_{\langle f_j|\Psi\rangle} \otimes |f_j\rangle$, and calculate ρ_B explicitly,

$$\rho_B = \mathrm{tr}_A\big(\underbrace{|\Psi\rangle\langle\Psi|}_{=\sum_{j,k} |\phi_j\rangle\langle\phi_k| \otimes |f_j\rangle\langle f_k|}\big) = \sum_{j,k} \underbrace{\mathrm{tr}(|\phi_j\rangle\langle\phi_k|)}_{\langle\phi_j|\phi_k\rangle} |f_j\rangle\langle f_k|$$

$$\rho_B = \sum_{j,k} \langle\phi_j|\phi_k\rangle |f_j\rangle\langle f_k| \Rightarrow \langle\phi_j|\phi_k\rangle = \lambda_j \delta_{jk}$$

Introducing orthonormal vectors $|e_j\rangle = |\phi_j\rangle/\sqrt{\lambda_j}$ for $\lambda_j \neq 0$, we have

$$|\Psi\rangle = \sum_j \sqrt{\lambda_j}\, |e_j\rangle \otimes |f_j\rangle .$$

<u>Purifications.</u> $|\Psi\rangle$ is a purification of ρ_A if

$$\rho_A = \mathrm{tr}_B\left(|\Psi\rangle\langle\Psi|\right).$$

What's the point?

① ρ_A can arise from an entangled bipartite pure state

② A purification can be a useful analytical tool, replacing a mixed-state analysis with a pure-state analysis on a larger space

THE CHURCH OF THE
LARGER HILBERT SPACE

Ensemble decomposition $\rho_A = \sum_\alpha p_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$

Purification $|\Psi\rangle = \underset{\substack{\uparrow \\ \text{relative-state} \\ \text{decomposition}}}{\sum_\alpha} \sqrt{p_\alpha}\,|\psi_\alpha\rangle \otimes \underset{\substack{\uparrow \\ \text{orthonormal vectors} \\ \text{in } \mathcal{H}_B}}{|f_\alpha\rangle}$

Putting it all together: purifications, ensemble decompositions, relative states, and Neumark extensions

VEC'ing an operator: VEC maps operator A on system S to a vector

$$|A) = |\Phi_A\rangle \equiv 1\otimes A|\tilde{\Phi}\rangle = \sum_{j=1}^{D}|f_j\rangle\otimes A|e_j\rangle = \sum_{j,k}|f_j\rangle\otimes|e_k\rangle\underbrace{\langle e_k|A|e_j\rangle}_{A_{kj}}$$

$$\sum_{j=1}^{D}|f_j\rangle\otimes|e_j\rangle \longleftarrow$$ unnormalized maximally entangled state

$|e_j\rangle$ ← arbitrary orthonormal basis for S

$|f_j\rangle$ ← arbitrary orthonormal basis for reference system R of dimension D

Recover A from $|A)$ via $\langle f_j, e_k|A) = A_{kj}$ VEC is 1-1 and onto

$$|A)(B| = \sum_{j,k}|f_j\rangle\langle f_k|\otimes A|e_j\rangle\langle e_k|B^\dagger$$

VEC preserves inner products: $(A|B) = \mathrm{tr}(A^\dagger B)$

$$\mathrm{tr}_R(|A)(A|) = \sum_j A|e_j\rangle\langle e_j|A^\dagger = AA^\dagger \longleftarrow$$ $|A)$ is a "purification" of $AA^\dagger$

$$\mathrm{tr}_S(|A)(A|) = \sum_{j,k}|f_j\rangle\langle f_k|\langle e_k|A^\dagger A|e_j\rangle = (A^\dagger A)^T \text{ in } R$$

T ↑ in fiducial basis $|f_j\rangle$

<u>Note</u>: $|A) = 1\otimes A|\tilde{\Phi}\rangle = \sum_{j,k}|f_j\rangle\otimes|e_k\rangle\langle e_k|A|e_j\rangle$

$$= \sum_{j,k}|f_j\rangle\langle f_j|A^T|f_k\rangle\otimes|e_k\rangle = A^T\otimes 1|\tilde{\Phi}_k\rangle$$

<u>Note</u>: SVD of $A_{kj} = (V\overset{\uparrow}{\Lambda}W^\dagger)_{kj}$ gives Schmidt decomposition of $|A)$.

choose diagonal in $|e_j\rangle$

$$|A) = \sum_{j,k}|f_j\rangle\otimes\underbrace{V\Lambda W^\dagger|e_j\rangle}_{\lambda_j V|e_j\rangle = \lambda_j|f_j\rangle} = \sum_j\lambda_j|f_j\rangle\otimes|f_j\rangle$$

Consider a positive operator $G = \sum_j \lambda_j |g_j\rangle\langle g_j|$ (G is a density operator if $\mathrm{tr}(G) = 1$).

rank of G

$$\underbrace{|\sqrt{G}\rangle}_{\text{“fiducial purification” of } G} = |\Phi_{\sqrt{G}}\rangle = 1 \otimes \underbrace{\sqrt{G}}_{G\text{-distortion}} |\tilde{\Phi}\rangle = \sum_{j=1}^{r} |f_j\rangle \otimes \sqrt{G}\,|g_j\rangle = \sum_{j=1}^{r} \sqrt{\lambda_j}\,|f_j\rangle \otimes |g_j\rangle$$

$$|\tilde{\Phi}\rangle = \sum_{j=1}^{d} |f_j\rangle \otimes |g_j\rangle$$

Extend R to arbitrary dimension N. By the Schmidt decomposition, any two "purifications" of G differ only by the orthogonal Schmidt vectors in R and thus are related by a unitary in R.

$$|\Phi\rangle = \mathcal{U} \otimes 1\, |\sqrt{G}\rangle = \sum_{j=1}^{r} \mathcal{U}|f_j\rangle \otimes \sqrt{G}\,|g_j\rangle$$

$$= \sum_{j=1}^{r} \sum_{\alpha=1}^{N} |f_\alpha\rangle \underbrace{\langle f_\alpha|\mathcal{U}|f_j\rangle}_{\mathcal{U}_{\alpha j}} \otimes \sqrt{G}\,|g_j\rangle$$

$$= \sum_{\alpha=1}^{N} |f_\alpha\rangle \otimes \underbrace{\sqrt{G} \sum_{j=1}^{r} \mathcal{U}_{\alpha j} |g_j\rangle}$$

$$= \sqrt{G} \sum_{\beta=1}^{N} \mathcal{U}_{\alpha\beta} |g_\beta\rangle$$

Extend $\mathcal{U}_{\alpha j}$ to an $N \times N$ unitary matrix $\mathcal{U}_{\alpha\beta}$.

Tack on additional basis states in S ($\sqrt{G}\,|g_\beta\rangle = 0$ for $\beta > r$)

$$= \sqrt{G} \sum_{\beta=1}^{N} |g_\beta\rangle\langle g_\beta| \mathcal{U}^T |e_\alpha\rangle$$

$$= \sqrt{G}\, \mathcal{U}^T |g_\alpha\rangle$$

$$= 1 \otimes \sqrt{G}\, \mathcal{U}^T \underbrace{\sum_{\alpha=1}^{N} |f_\alpha\rangle \otimes |g_\alpha\rangle}_{|\tilde{\Phi}\rangle}$$

We end up with several ways to write an arbitrary purification of G:

arbitrariness is in $\mathcal{U}$

$$|\bar{\Phi}\rangle = \mathcal{U}\otimes 1\,|\sqrt{G}\rangle = \mathcal{U}\otimes\sqrt{G}\,|\bar{\Psi}\rangle$$

$$= 1\otimes\sqrt{G}\,\mathcal{U}^T\,|\tilde{\bar{\Psi}}\rangle$$

relative-state decomposition

$$= \sum_{\alpha=1}^{N} |f_\alpha\rangle\otimes|\bar{\tilde{\psi}}_\alpha\rangle$$

$$|\bar{\tilde{\psi}}_\alpha\rangle = \sqrt{G}\,\mathcal{U}^T|g_\alpha\rangle = \sum_{j=1}^{r} \mathcal{U}_{\alpha j}\sqrt{G}\,|g_j\rangle = \sum_{j=1}^{r} \mathcal{U}_{\alpha j}\sqrt{\lambda_j}\,|g_j\rangle$$

are a decomposition of G.

Purifications and ensemble decompositions are in 1-1 correspondence, both specified by $N\times r$ partial unitary matrix $\mathcal{U}_{\alpha j}$.

HJW theorem

$$= \sum_{\alpha=1}^{N} |f_\alpha\rangle\otimes\sqrt{G}\,|\bar{\phi}_\alpha\rangle$$

$|\bar{\phi}_\alpha\rangle = \sum_{j=1}^{r} \mathcal{U}_{\alpha j}|g_j\rangle$ resolve the support of G (projector $P_G = \sum_{j=1}^{r}|g_j\rangle\langle g_j|$

$$= \sum_{\alpha=1}^{N} |f_\alpha\rangle\otimes\sqrt{G}\,\overbrace{\mathcal{U}^T|g_\alpha\rangle}$$

orthonormal vectors in an extended space; called the Newmark extension of $|\bar{\phi}_\alpha\rangle$

$$P_G\,\mathcal{U}^T|g_\alpha\rangle = \sum_j |g_j\rangle\,\mathcal{U}_{\alpha j} = |\bar{\phi}_\alpha\rangle$$

The unitary matrix in a change between ensemble decompositions finds another home as a unitary operator $\mathcal{U}$ on R, which transforms between purifications, or as a unitary operator $\mathcal{U}^T$ on S, which acts in the Newmark extension.