Quantum information theory

Lectures 24 – 25

Qubit operations

Qubit operations: consider only trace-preserving operations

① Count parameters: $\underset{\substack{\uparrow \\ D^*}}{16} - \underset{\substack{\uparrow \\ t\text{-}p}}{4} = 12$ parameters

② An operation is specified by its action on pure states. We only need to say what happens to the surface of the Bloch sphere.

$$\mathcal{Q}(I) = aI + \vec{c}\cdot\vec{\sigma}$$

$$\mathcal{Q}(\sigma_j) = b_j I + \sum_k \sigma_k M_{kj}$$

$$\Rightarrow \mathcal{Q}(\rho) = \mathcal{Q}\left(\tfrac{1}{2}(I + \vec{S}\cdot\vec{\sigma})\right)$$

$$= \frac{1}{2}\left(\underbrace{(a + \vec{S}\cdot\vec{b})}_{t\text{-}p \Rightarrow \vec{b}=0 \text{ and } a=1} I + \vec{c}\cdot\vec{\sigma} + \sum_{j,k} \underbrace{\sigma_k M_{kj} S_j}_{M\vec{S}\cdot\vec{\sigma} = \vec{S}\cdot M^T\vec{\sigma}}\right)$$

Summary:

$$\mathcal{Q}(I) = I + \vec{c}\cdot\vec{\sigma} \quad \leftarrow \text{3 parameters}$$

$$\mathcal{Q}(\sigma_j) = \sum_k \sigma_k M_{kj} \quad \leftarrow \text{9 parameters}$$

$$\mathcal{Q}(\rho) = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\underbrace{(M\vec{S} + \vec{c})}_{\vec{S}'}\right)$$

$\leftarrow$ affine transformation of the Bloch vector

$$\vec{S}' = M\vec{S} + \underset{\substack{\uparrow \\ \text{displacement}}}{\vec{c}}$$

Look at M more closely.

Polar decomposition:

$$M = O\sqrt{M^T M}$$

$\sqrt{M^T M}$ — real symmetric positive matrix

$$\sqrt{M^T M} = R_1^T D R_1$$

R_1 — rotation

D — diagonal with positive entries

O — orthogonal matrix: $\det O = \pm 1$

$$O = R\,P$$

P — identity $(\det P = +1)$ or a reflection or inversion $(\det P = -1)$; we can choose a standard reflection, say thru x-y plane, but prefer to leave it general

R — rotation: $\det R = +1$

$$M = R P R_1^T D R_1 = \underbrace{R R_3}_{R_2} \overbrace{P D}^{T} R_1$$

T — diagonal matrix, with entries of either sign

R_2 (another rotation)

$$P R_1^T = P R_1^T \underbrace{P P}_{I} = R_3 P$$

R_3 — rotation

Summarize: $M = R_2 T R_1$ $\qquad T = \begin{pmatrix} t_x & 0 & 0 \\ 0 & t_y & 0 \\ 0 & 0 & t_z \end{pmatrix}$

R_2: 3 parameters $\qquad R_1$: 3 parameters

$T_{jk} = t_j \delta_{jk}$ $\qquad \begin{pmatrix} \text{contraction along 3 coördinate axes} \end{pmatrix} \times \begin{pmatrix} \text{identity } \underline{\text{OR}} \text{ a reflection or inversion} \end{pmatrix}$

3 parameters

$$\vec{S}' = R_2 T R_1 \vec{S} + \vec{c} = R_2 (T R_1 \vec{S} + \vec{d}), \quad \vec{d} = R_2^T \vec{c}$$

$\vec{d}$ — displacement of center of Bloch sphere

How does this look in terms of the quantum operation? Define a new operation that only has the contraction and displacement in it,

$\rho = \frac{1}{2}(I + \vec{S}\cdot\vec{\sigma})$

$$\mathcal{B}(\rho) = \frac{1}{2}\left(I + \vec{\sigma}\cdot\left(T\vec{S} + \vec{d}\right)\right).$$

$\mathcal{B}(I) = I + \vec{\sigma}\cdot\vec{d}$

$\mathcal{B}(X) = t_x X$

$\mathcal{B}(Y) = t_z Y$

$\mathcal{B}(Z) = t_z Z$

We claim that

$$\mathcal{Q}(\rho) = \underset{\substack{\uparrow \\ \text{final unitary}}}{U_{R_2}}\, \mathcal{B}\big(\underset{\substack{\uparrow \\ \text{initial unitary}}}{U_{R_1}}\rho U_{R_1}^{\dagger}\big)\, U_{R_2}^{\dagger}$$

$$\mathcal{B} \leftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ d_x & t_x & 0 & 0 \\ d_y & 0 & t_y & 0 \\ d_z & 0 & 0 & t_z \end{pmatrix}$$

Check:

$$U_{R_1}\rho U_{R_1}^{\dagger} = \frac{1}{2}\big(1 + \underbrace{U_{R_1}\vec{\sigma}U_{R_1}^{\dagger}\cdot\vec{S}}_{R_1^T\vec{\sigma}\cdot\vec{S} = \vec{\sigma}\cdot R_1\vec{S}}\big) = \frac{1}{2}\left(1 + \vec{\sigma}\cdot R_1\vec{S}\right).$$

$$\mathcal{B}(U_{R_1}\rho U_{R_1}^{\dagger}) = \frac{1}{2}\left(1 + \vec{\sigma}\cdot\left(TR_1\vec{S} + \vec{d}\right)\right)$$

$$U_{R_2}\mathcal{B}(U_{R_1}\rho U_{R_1}^{\dagger})U_{R_2}^{\dagger} = \frac{1}{2}\Big(1 + \underbrace{U_{R_2}\vec{\sigma}U_{R_2}^{\dagger}\cdot\left(TR_1\vec{S} + \vec{d}\right)}\Big)$$

$$= R_2^T\vec{\sigma}\cdot\left(TR_1\vec{S} + \vec{d}\right)$$

$$= \vec{\sigma}\cdot\left(R_2TR_1\vec{S} + R_2\vec{d}\right)$$

$$= \vec{\sigma}\cdot\left(M\vec{S} + \vec{c}\right)$$

$$= \frac{1}{2}\left(1 + \vec{\sigma}\cdot\left(M\vec{S} + \vec{c}\right)\right)$$

$$= \mathcal{Q}(\rho)$$

Ⓐ

Complete positivity:

① $\frac{1}{2}\mathrm{tr}\left(\sigma_\alpha^\dagger \mathcal{B}(\sigma_\beta)\right) = \mathcal{B}_{\alpha\beta} \longrightarrow$ $\mathcal{B}_{00} = 1$, $\mathcal{B}_{j0} = d_j$, $\mathcal{B}_{0j} = 0$, $\mathcal{B}_{kj} = T_{kj} = t_j \delta_{jk}$

This matrix is our present characterization of $\mathcal{B}$.

$\mathcal{B}_{\alpha\beta}$ ← Choi matrix

$$= \frac{1}{2}(\sigma_\alpha|\mathcal{B}^\#|\sigma_\beta)$$

$$\Rightarrow \mathcal{B}^\# = \sum_{\alpha,\beta} \frac{1}{2}|\sigma_\alpha)\, \frac{1}{2}(\sigma_\alpha|\mathcal{B}^\#|\sigma_\beta)(\sigma_\beta| = \frac{1}{2}\sum_{\alpha,\beta} \mathcal{B}_{\alpha\beta}\, \sigma_\alpha \odot \sigma_\beta^\dagger$$

② For complete positivity, we require that

$$\frac{1}{2}(\sigma_\alpha|\mathcal{B}|\sigma_\beta) = \frac{1}{2}\mathrm{tr}\left(\sigma_\alpha^\dagger \mathcal{B}^\#(\sigma_\beta)\right) = \mathcal{B}^\#_{\alpha\beta} \equiv \chi_{\alpha\beta}$$

be a positive matrix. ($\chi_{\alpha\beta}$ ← process matrix)

$$\mathcal{B}^\#_{\alpha\beta} = \frac{1}{2}\mathrm{tr}\left(\sigma_\alpha^\dagger \mathcal{B}^\#(\sigma_\beta)\right) = \frac{1}{4}\underbrace{\sum_{\gamma,\delta} \mathcal{B}_{\gamma\delta}\,\mathrm{tr}\left(\sigma_\alpha \sigma_\gamma \sigma_\beta \sigma_\delta\right)}$$

Choi transformation

$\mathcal{B}^\#_{\alpha\beta} = \chi_{\alpha\beta}$ ← process matrix

CP ⟺ this matrix has nonnegative eigenvalues

$$\mathcal{B} = \frac{1}{2}\sum_{\alpha,\beta} \chi_{\alpha\beta}\, \sigma_\alpha \odot \sigma_\beta^\dagger = \frac{1}{2}\sum_{\alpha,\beta} \chi_{\alpha\beta} |\sigma_\alpha)(\sigma_\beta|$$

↑ Right-left diagonalizing this superoperator gives orthogonal Kraus operators.

Positivity: $|\vec{S}'| = |M\vec{S} + \vec{c}| \le 1$ for all $\vec{S}$

Kraus decompositions] → What happened to these?
Complete positivity] → See insert (A)

Only need to worry about $\mathcal{B}$ – contractions and displacement

Examples: $\vec{d} = 0$

(1) $T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is cp

projection of Bloch sphere onto z axis

Kraus operators: P_0, P_1 ; $Q = P_0 \odot P_0 + P_1 \odot P_1$

$$Q(X) = P_0 X P_0 + P_1 X P_1 = 0$$
$$Q(Y) = P_0 Y P_0 + P_1 Y P_1 = 0 \Rightarrow T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$Q(Z) = P_0 Z P_0 + P_1 Z P_1 = Z$$

This T corresponds to measurement in the Z basis.

(2) $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is <u>not</u> cp

projection of Bloch sphere onto equatorial plane

Try $Q \equiv \frac{3}{4} 1 \odot 1 + \frac{1}{4} X \odot X + \frac{1}{4} Y \odot Y - \frac{1}{4} Z \odot Z$

$Q(1) = 1$, $Q(X) = X$, $Q(Y) = Y$, $Q(Z) = 0$

$\Rightarrow T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

These are the left-right eigenvalues of Q: One is negative, so Q is not cp.

Interesting qubit operations:

1. Stochastic flip operations:

Kraus operators: $\sqrt{p}\,\mathbb{1}$, $\sqrt{1-p}\,\sigma_j$,

$$\mathcal{Q} = p\,\underbrace{\mathbb{1}\odot\mathbb{1}}_{\mathcal{I}} + (1-p)\underbrace{\sigma_j\odot\sigma_j}_{\text{flip}}$$

Probability p for nothing to happen; probability $1-p$ for flip

$$\mathcal{Q}(\rho) = p\rho + (1-p)\sigma_j\rho\sigma_j$$

$\sigma_j = X \leftrightarrow$ bit flip

$\sigma_j = Z \leftrightarrow$ phase flip

$\sigma_j = Y = -iZX = iXZ$

$\leftrightarrow$ bit-phase flip

Bloch-sphere description: Bit flip

$$\mathcal{Q} = p\,\mathbb{1}\odot\mathbb{1} + (1-p)X\odot X \qquad \mathcal{Q}(\mathbb{1}) = \mathbb{1} \Rightarrow \vec{c} = 0$$

$$\begin{aligned}\mathcal{Q}(X) &= X\\ \mathcal{Q}(Y) &= (2p-1)Y\\ \mathcal{Q}(Z) &= (2p-1)Z\end{aligned} \quad\longleftrightarrow\quad T = \begin{pmatrix}1 & 0 & 0\\ 0 & 2p-1 & 0\\ 0 & 0 & 2p-1\end{pmatrix}$$

contraction along y and z axes

$p=1$: identity

$p=1/2$: measurement in X basis

$p=0$: pure bit flip

2. Depolarizing operation:

Kraus operators: $\frac{\sqrt{p}}{2}\mathbb{1}$, $\frac{\sqrt{p}}{2}X$, $\frac{\sqrt{p}}{2}Y$, $\frac{\sqrt{p}}{2}Z$, $\sqrt{1-p}\,\mathbb{1}$

$$\mathcal{Q} = p\,\underbrace{\frac{1}{4}\left(\mathbb{1}\odot\mathbb{1} + X\odot X + Y\odot Y + Z\odot Z\right)}_{} + (1-p)\,\mathbb{1}\odot\mathbb{1}$$

<u>completely</u> depolarizing operation: $\rho \to \frac{1}{2}\mathbb{1}$

$= \mathbb{I}/2$ (generally $\mathbb{I}/D$: $\frac{1}{D}\mathbb{I}(\rho) = \mathbb{1}/D$)

left-right identity superoperator

$$\mathcal{E}(\rho) = p\,\tfrac{1}{2}\mathbb{1} + (1-p)\,\rho$$

Probability p for complete depolarization; probability $1-p$ for nothing

Bloch sphere description: $\mathcal{E}(\mathbb{1}) = \mathbb{1} \Rightarrow \vec{c} = 0$

$$\mathcal{E}(\sigma_j) = (1-p)\,\sigma_j \longleftrightarrow T = (1-p)\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

uniform contraction

③ Amplitude damping (spontaneous decay from $|0\rangle$ to $|1\rangle$):

Measurement model:

$$\mathcal{U}: \quad |1\rangle\left|\begin{matrix}\text{environment}\\ \text{ground state}\end{matrix}\right\rangle \longrightarrow |1\rangle\left|\begin{matrix}\text{environment}\\ \text{ground state}\end{matrix}\right\rangle$$

$$|0\rangle\left|\begin{matrix}\text{environment}\\ \text{ground state}\end{matrix}\right\rangle \longrightarrow \sqrt{1-\gamma}\,|0\rangle\boxed{\left|\begin{matrix}\text{environment}\\ \text{ground state}\end{matrix}\right\rangle} + \underbrace{\sqrt{\gamma}}_{\text{amplitude for decay}}\,|1\rangle\boxed{\left|\begin{matrix}\text{decay}\\ \text{products}\end{matrix}\right\rangle}$$

orthogonal

$$\begin{aligned}
\rho\otimes|egs\rangle\langle egs| = \;& \rho_{11}\,|1\rangle|egs\rangle\langle 1|\langle egs| \\
&+ \rho_{10}\,|1\rangle|egs\rangle\langle 0|\langle egs| \\
&+ \rho_{01}\,|0\rangle|egs\rangle\langle 1|\langle egs| \\
&+ \rho_{00}\,|0\rangle|egs\rangle\langle 0|\langle egs|
\end{aligned}$$

$$\begin{aligned}
\xrightarrow[\mathcal{U}]{}\; & \rho_{11}\,|1\rangle|egs\rangle\langle 1|\langle egs| \\
&+ \rho_{10}\,|1\rangle|egs\rangle\left(\sqrt{1-\gamma}\,\langle 0|\langle egs| + \sqrt{\gamma}\,\langle 1|\langle dp|\right) \\
&+ \rho_{01}\left(\sqrt{1-\gamma}\,|0\rangle|egs\rangle + \sqrt{\gamma}\,|1\rangle|dp\rangle\right)\langle 1|\langle egs| \\
&+ \rho_{00}\left(\sqrt{1-\gamma}\,|0\rangle|egs\rangle + \sqrt{\gamma}\,|1\rangle|dp\rangle\right)\left(\sqrt{1-\gamma}\,\langle 0|\langle egs| + \sqrt{\gamma}\,\langle 1|\langle dp|\right)
\end{aligned}$$

$\mathcal{Q}(\rho)$ is obtained by tracing out the environment:

$$\mathcal{Q}(\rho) = \rho_{11}|1\rangle\langle 1| + \sqrt{1-\gamma}\,\rho_{10}|1\rangle\langle 0| + \sqrt{1-\gamma}\,\rho_{01}|0\rangle\langle 1|$$
$$+ \rho_{00}\big((1-\gamma)|0\rangle\langle 0| + \gamma|1\rangle\langle 1|\big)$$

$$= \left(|1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0|\right)\rho\left(|1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0|\right)$$
$$+ \sqrt{\gamma}\,|1\rangle\langle 0|\,\rho\,|0\rangle\langle 1|\sqrt{\gamma}$$

Kraus operators: $A_1 = |1\rangle\langle 1| + \sqrt{1-\gamma}|0\rangle\langle 0| \longleftrightarrow \begin{pmatrix} \sqrt{1-\gamma} & 0 \\ 0 & 1 \end{pmatrix}$

$A_2 = \sqrt{\gamma}\,|1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{pmatrix}$

$$\mathcal{Q} = A_1 \odot A_1^\dagger + A_2 \odot A_2^\dagger$$

Bloch-sphere description:

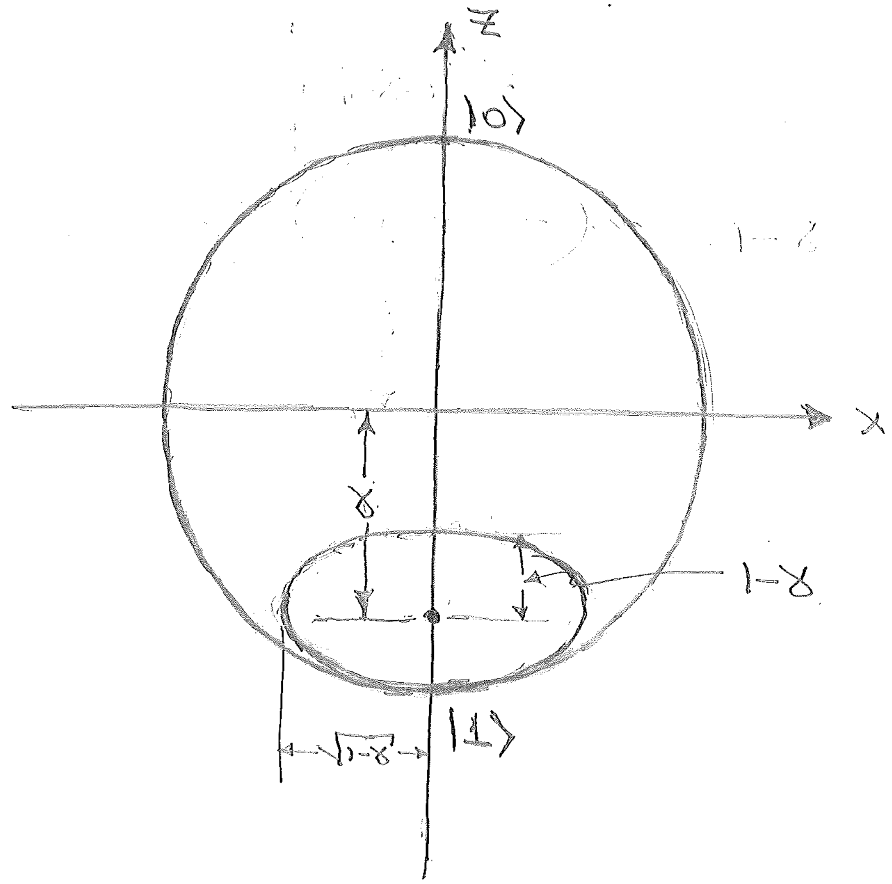
$$\mathcal{Q}(1) = \underbrace{A_1 A_1^\dagger}_{\begin{pmatrix} 1-\gamma & 0 \\ 0 & 1 \end{pmatrix}} + \underbrace{A_2 A_2^\dagger}_{\begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}} = \begin{pmatrix} 1-\gamma & 0 \\ 0 & 1+\gamma \end{pmatrix} = 1 - \gamma Z \Rightarrow \vec{c} = -\gamma \vec{e}_z$$

$$\mathcal{Q}(X) = \underbrace{A_1 X A_1^\dagger}_{\sqrt{1-\gamma}\,X} + \underbrace{A_2 X A_2^\dagger}_{0} = \sqrt{1-\gamma}\,X$$

$$\longrightarrow T = \begin{pmatrix} \sqrt{1-\gamma} & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}$$

$$\mathcal{Q}(Y) = \sqrt{1-\gamma}\,Y$$

$$\mathcal{Q}(Z) = \underbrace{A_1 Z A_1^\dagger}_{\begin{pmatrix} 1-\gamma & 0 \\ 0 & -1 \end{pmatrix}} + \underbrace{A_2 Z A_2^\dagger}_{\begin{pmatrix} 0 & 0 \\ 0 & \gamma \end{pmatrix}} = \begin{pmatrix} 1-\gamma & 0 \\ 0 & -(1-\gamma) \end{pmatrix} = (1-\gamma) Z$$



Amplitude damping

Spontaneous decay to |1⟩

This is called amplitude damping because it damps the amplitude to be in |0⟩.

Convert to a differential equation that describes decay:

$\gamma = R\,dt$

$$\vec{S}' = T\vec{S} + \vec{C} = \begin{pmatrix} 1-\frac{1}{2}R\,dt & & \\ & 1-\frac{1}{2}R\,dt & \\ & & 1-R\,dt \end{pmatrix}\begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix} - R\,dt\,\hat{z}$$

$$\frac{dS_x}{dt} = -\frac{1}{2}R S_x, \quad \frac{dS_y}{dt} = -\frac{1}{2}R S_y, \quad \frac{dS_z}{dt} = -R S_z - R$$

$$S_x = e^{-Rt/2} S_x(0) \quad S_y(t) = e^{-Rt/2} S_y(0) \quad S_z(t) = -1 + e^{-Rt}(1 + S_z(0))$$