Quantum information theory

Lectures 5-6

Linear algebra and axioms of quantum mechanics

The behavior of quantum systems is described in a complex vector space with an inner product.

$$|\psi\rangle = \sum_{j=1}^{D} c_j |\psi_j\rangle$$

dimension of the space (D)

$|\psi\rangle$: vector "ket"

$|\psi_j\rangle$: basis of linearly independent vectors

A complex inner-product vector space is called a <u>Hilbert Space</u>.

<u>Inner product</u>: $(|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle$ ← definition of "bra" $\langle\phi|$ as a linear function from kets to $\mathbb{C}$.

① Linear in $|\psi\rangle$: $(|\phi\rangle, a|\psi\rangle + b|\chi\rangle) = a(|\phi\rangle, |\psi\rangle) + b(|\phi\rangle, |\chi\rangle)$

② Complex symmetric: $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$ ← complex conjugate

$\Rightarrow$ <u>antilinearity</u> in $|\phi\rangle$: $(a|\phi\rangle + b|\chi\rangle, |\psi\rangle) = a^*(|\phi\rangle, |\psi\rangle) + b^*(|\chi\rangle, |\psi\rangle)$

③ $(|\psi\rangle, |\psi\rangle) \geq 0$, with equality iff $|\psi\rangle = 0$

$$|\phi\rangle \longleftrightarrow \langle\phi|$$

$$a|\phi\rangle + b|\chi\rangle \longleftrightarrow a^*\langle\phi| + b^*\langle\chi|$$

Often write $|e_j\rangle = |j\rangle$

Orthonormal basis $|e_j\rangle$, $j = 1, \dots, D$: $\langle e_j|e_k\rangle = \delta_{jk}$

$$|\psi\rangle = \sum_j c_j |e_j\rangle = \sum_j |e_j\rangle\langle e_j|\psi\rangle$$

$c_j = \langle e_j|\psi\rangle$

$$|\phi\rangle = \sum_j d_j |e_j\rangle = \sum_j |e_j\rangle\langle e_j|\phi\rangle$$

$$\langle\phi|\psi\rangle = \sum_j d_j^* c_j = \sum_j \langle\phi|e_j\rangle\langle e_j|\psi\rangle$$

$|\psi\rangle \longleftrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$: representation

$\langle\phi| = \sum_j d_j^* \langle e_j|$

$\longleftrightarrow (d_1^* \dots d_D^*)$

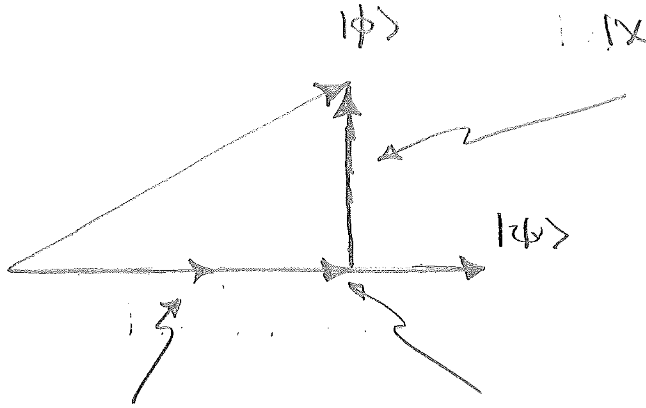
$\langle\phi|\psi\rangle = (d_1^* \dots d_D^*)\begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$

Shwarz inequality:

$$|\langle\phi|\psi\rangle| \le \langle\phi|\phi\rangle^{1/2}\langle\psi|\psi\rangle^{1/2}$$

Equality iff $|\phi\rangle = \alpha|\psi\rangle$

$$|\chi\rangle = |\phi\rangle - |\xi\rangle = |\phi\rangle - \frac{|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$$



The Schwarz inequality is simply the statement that this vector has nonnegative length, i.e., $\langle\chi|\chi\rangle \ge 0$.

$|e\rangle \equiv |\psi\rangle/\sqrt{\langle\psi|\psi\rangle}$ $\quad |\xi\rangle = |e\rangle\langle e|\phi\rangle = \dfrac{|\psi\rangle\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$

$\langle e|e\rangle = 1$

$|\phi\rangle = |\chi\rangle + |\xi\rangle$, $\quad \langle\chi|\xi\rangle = 0$ (by construction) Check it!

$$\langle\phi|\phi\rangle = \langle\chi|\chi\rangle + \langle\xi|\xi\rangle \ge \langle\xi|\xi\rangle = \frac{|\langle\psi|\phi\rangle|^2}{\langle\psi|\psi\rangle}$$

Notice that this is just Pythagoras's theorem

$$\Longrightarrow \boxed{\langle\phi|\phi\rangle\langle\psi|\psi\rangle \ge |\langle\psi|\phi\rangle|^2}$$

Equality iff $|\chi\rangle = 0 \Longleftrightarrow |\phi\rangle = |\xi\rangle = |\psi\rangle\dfrac{\langle\psi|\phi\rangle}{\langle\psi|\psi\rangle}$

i.e., $|\phi\rangle = \alpha|\psi\rangle$

Linear operators: A

$\hookrightarrow A\left(a|\psi\rangle + b|\phi\rangle\right) = a\,A|\psi\rangle + b\,A|\phi\rangle$

Matrix elements $\langle e_j|A|e_k\rangle = A_{jk}$

representation

$$A \longleftrightarrow \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1D} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{D1} & & & A_{DD} \end{pmatrix}$$

$$A|\psi\rangle = \sum_j |e_j\rangle\langle e_j|A|\psi\rangle$$

$$= \sum_{j,k} |e_j\rangle\langle e_j|A|e_k\rangle\langle e_k|\psi\rangle$$

$$= \sum_{j,k} |e_j\rangle A_{jk} c_k$$

$$A|\psi\rangle \longleftrightarrow \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$$

Outer-product notation: $A = \sum_{j,k} |e_j\rangle A_{jk}\langle e_k|$

$$= \sum_{j,k} |e_j\rangle\langle e_j|A|e_k\rangle\langle e_k|$$

Unit operator: $1 = I = \sum_j |e_j\rangle\langle e_j|$ (resolution of the identity)

$$AB = \sum_{j,k,l} |e_j\rangle\langle e_j|A|e_l\rangle\langle e_l|B|e_k\rangle\langle e_k|$$

$$= \sum_{j,k} |e_j\rangle\langle e_k| \sum_l A_{jl}B_{lk}$$

$\langle\phi|A \leftrightarrow A^\dagger|\phi\rangle$; $A|\phi\rangle \leftrightarrow \langle\phi|A^\dagger$

Adjoint operator (Hermitian conjugate):

$$\left(|\phi\rangle, A|\psi\rangle\right) = \left(A^\dagger|\phi\rangle, |\psi\rangle\right) = \left(|\psi\rangle, A^\dagger|\phi\rangle\right)^*$$

$$\langle\phi|A|\psi\rangle = \qquad\qquad = \langle\psi|A^\dagger|\phi\rangle^*$$

no translation into Dirac notation

① Adjoint is antilinear: $(aA + bB)^\dagger = a^*A^\dagger + b^*B^\dagger$

② $(A^\dagger)_{jk} = \langle e_j|A^\dagger|e_k\rangle = \langle e_k|A|e_j\rangle^* = A^*_{kj} \Longrightarrow (A^\dagger)^\dagger = A$

③ $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi| \Rightarrow$

④ $(AB)^\dagger = B^\dagger A^\dagger$

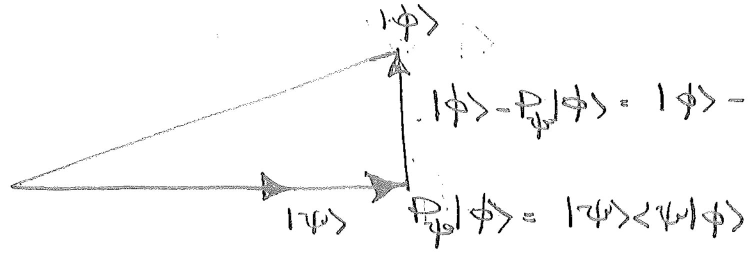
$A = \sum_{jk} |e_j\rangle\langle e_k| A_{jk}$

$A^\dagger = \sum_{jk} |e_k\rangle\langle e_j| A^*_{jk} = \sum_{jk} |e_j\rangle\langle e_k| A^*_{kj}$

(equivalent to ②)

__Projection operators:__

1-d projector $P_\psi = |\psi\rangle\langle\psi|$, $\langle\psi|\psi\rangle = 1$



multi-dimensional projector: Let S be a d-dimensional subspace spanned by orthonormal vectors $|e_j\rangle$, $j=1,\dots,d$. The projection operator onto S is

$$P_S = \sum_{j=1}^{d} |e_j\rangle\langle e_j| \longleftarrow$$ unit operator in the subspace S

An operator has a __spectral__ decomposition (or __eigendecomposition__) if it can be written as

$$A = \sum_j \lambda_j \overbrace{|e_j\rangle\langle e_j|}^{P_j} = \sum_\lambda \lambda P_\lambda \qquad 1 = \sum_j P_j = \sum_\lambda P_\lambda$$

orthonormal __eigenvectors__

$A|e_j\rangle = \lambda_j |e_j\rangle$

__eigenvalues__

All vectors in S_λ are eigenvectors with eigenvalue λ; they are said to be __degenerate__. Eigenvectors with different eigenvalues are orthogonal; they are nondegenerate.

__Support__ S is the subspace spanned by eigenvectors with nonzero eigenvalue. __Rank__ is the dimension of S. __Null subspace__ (kernel) K is subspace spanned by eigenvectors with zero eigenvalue. S and K are orthogonal.

$P_\lambda = \sum_{j:\lambda_j=\lambda} |e_j\rangle\langle e_j|$ is the projector onto the subspace S_λ spanned by eigenvectors with eigenvalue λ; __eigenprojector__ onto __eigensubspace__ S_λ

Commutator: $[A,B] = AB - BA$

Normal operator M: $[M, M^\dagger] = 0$

Theorem. An operator has a spectral decomposition iff it is normal.

Proof: s.d. $\Rightarrow$ normal (this is trivial)

s.d. $\Leftarrow$ normal (N&C Box 2.2)

> The eigenvalues and eigenvectors of nonnormal operators are complicated. They make up a major topic in linear algebra, which we don't have to worry about.

Hermitian operators: $H = H^\dagger$

Hermitian operators are normal operators whose eigenvalues are real.

Unitary operators: $U^\dagger U = 1 = UU^\dagger$

> Only need $U^\dagger U = 1$ in finite dimensions.
> $U^\dagger U = 1 \Rightarrow (UU^\dagger)^2 = UU^\dagger$
> $\Rightarrow UU^\dagger = P$
> $tr(P) = tr(UU^\dagger) = tr(U^\dagger U) = tr(I) = D$
> $\Rightarrow I = P = UU^\dagger$

Unitary operators are normal operators whose eigenvalues are phases.

① U preserves inner products:

$$(\langle\psi|U^\dagger)(U|\phi\rangle) = \langle\psi|U^\dagger U|\phi\rangle = \langle\psi|\phi\rangle$$

$\Rightarrow$ ② U takes orthonormal bases to orthonormal bases:

Basis change $\longrightarrow$ $|f_j\rangle = U|e_j\rangle \iff U = \sum_j |f_j\rangle\langle e_j|$

$$= \sum_k |e_k\rangle\langle e_k|U|e_j\rangle$$

$$= \sum_k |e_k\rangle U_{kj} \longrightarrow \quad = \sum_k |e_k\rangle\langle e_j| U_{kj}$$

(unitary) transformation matrix between the bases.

$$UU^\dagger = 1 \iff \delta_{jk} = \sum_l U_{jl}(U^\dagger)_{lk} = \sum_l U_{jl} U^*_{kl}$$

> rows and columns of U_{jk} are orthonormal vectors

Projection operators are Hermitian operators all of whose eigenvalues are 0 or 1. A Hermitian operator P is a projection operator iff $P^2 = P$.

Operator functions: A function $f: \mathbb{C} \to \mathbb{C}$ can be extended to a function on <u>normal</u> operators by

$$f(A) = f\left(\sum_j \lambda_j |e_j\rangle\langle e_j|\right) = \sum_j f(\lambda_j) |e_j\rangle\langle e_j|$$

<u>Example:</u> $U = \sum_j e^{i\phi_j} |e_j\rangle\langle e_j|$. Define a Hermitian operator $H = \sum_j \phi_j |e_j\rangle\langle e_j|$. Then

$$U = \sum_j e^{i\phi_j} |e_j\rangle\langle e_j| = e^{iH} \longleftarrow$$

Any unitary can be written like this for some $H = H^\dagger$

Trace: For any operator A,

$$\begin{pmatrix}\text{trace}\\ \text{of } A\end{pmatrix} = tr(A) \equiv \sum_j \langle e_j|A|e_j\rangle$$

① tr is linear: $tr(aA + bB) = a\,tr(A) + b\,tr(B)$

② Basis-independence:

$$\sum_j \langle e_j|A|e_j\rangle = \sum_{j,k} \langle e_j|f_k\rangle\langle f_k|A|e_j\rangle$$

insert resolution of 1

$$= \sum_{j,k} \langle f_k|A|e_j\rangle\langle e_j|f_k\rangle$$

$$= \sum_k \langle f_k|A|f_k\rangle$$

remove resolution of 1

3. $\mathrm{tr}(AB) = \sum_{j,k} \langle e_j | A | e_k \rangle \langle e_k | B | e_j \rangle$ Insert and remove resolution of 1

$= \sum_{j,k} \langle e_k | B | e_j \rangle \langle e_j | A | e_k \rangle$

$= \mathrm{tr}(BA)$

4. Cyclic property of trace: $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$

5. $\mathrm{tr}(|\psi\rangle\langle\phi|) = \sum_j \langle e_j | \psi \rangle \langle \phi | e_j \rangle$

$= \sum_j \langle \phi | e_j \rangle \langle e_j | \psi \rangle$

$= \langle \phi | \psi \rangle \longleftarrow$ remove resolution of 1

6. $\mathrm{tr}(A|\psi\rangle\langle\phi|) = \mathrm{tr}(|\psi\rangle\langle\phi|A) = \langle \phi | A | \psi \rangle$

$\boxed{\text{tr turns outer product into inner product}}$

<u>Simultaneous eigenvectors theorem.</u> Two normal operators have simultaneous eigenvectors iff they commute.

<u>Proof:</u>

$\Longrightarrow$ Trivial

$\Longleftarrow$ Let A and B be normal operators such that $[A,B]=0$. A has eigendecomposition

$$A = \sum_\lambda \lambda \underbrace{P_\lambda}_{\text{projector onto eigensubspace } S_\lambda}.$$

Any vector $|e\rangle$ in S_λ is an eigenvector of A with eigenvalue λ, i.e., $A|e\rangle = \lambda|e\rangle$.

$$A(B|e\rangle) \underset{[A,B]=0}{=} B(A|e\rangle) = \lambda B|e\rangle$$

So $B|e\rangle$ is also an eigenvector of A with eigenvalue λ and thus is in S_λ. This means B maps each eigensubspace of A into itself, so we can write it as $B = \sum_\lambda{}' B_\lambda$, where B_λ operates in S_λ. Diagonalizing the B_λ's gives eigenvectors of B that lie in the subspaces S_λ and thus are also eigenvectors of A.

Notice that if B-eigenvectors from different S_λ are degenerate, we can superpose them to make B-eigenvectors that do not lie in any S_λ.

"Axioms" of quantum mechanics.

1. <u>Quantum states.</u> The <u>state</u> of a quantum system is a <u>ray</u> in Hilbert space. A ray is the collection of states $a|\psi\rangle$, $a \in \mathbb{C}$ ($b|\psi\rangle$ is the same state as $|\psi\rangle$). The set of rays is called <u>projective Hilbert space.</u>

(a) We generally normalize states so that $\langle\psi|\psi\rangle = 1$, but it is still true that $e^{i\phi}|\psi\rangle$ is the same state as $|\psi\rangle$.
global phase vs. relative phase in a superposition

A normalized vector is called a <u>state vector.</u>

[(b) We can get rid of the global phase freedom by identifying states with 1-d projectors $|\psi\rangle\langle\psi|$.]

Pure states

2. <u>Observables.</u> An observable is a Hermitian operator $A = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_\lambda \lambda P_\lambda$. The result of a measurement of A is one of the eigenvalues λ. When the system is in state $|\psi\rangle$, the probability of getting eigenvalue λ is

$$p(\lambda) = \langle\psi|P_\lambda|\psi\rangle = \sum_{\lambda_j=\lambda} \underbrace{|\langle e_j|\psi\rangle|^2}_{\text{probability for eigenvector } |e_j\rangle}$$
$$= \mathrm{tr}(P_\lambda |\psi\rangle\langle\psi|)$$

$\boxed{\langle e_j|\psi\rangle = c_j \text{ is a probability amplitude}}$

Eigenvalues label the measurement results. Probabilities only depend on the eigenvectors. We often speak of a measurement "in the basis $|e_j\rangle$" and say the result is $|e_j\rangle$.

von Neumann measurement statistics

von Neumann measurement

3. Post-measurement states. A measurement with result $|e_j\rangle$ leaves the system in the state $|e_j\rangle$:

$$|\psi\rangle \longrightarrow \frac{P_j|\psi\rangle}{\langle\psi|P_j|\psi\rangle^{1/2}} = \frac{P_j|\psi\rangle}{p_j}$$

"collapse of the wave function"

Dynamics of an isolated system

4. Dynamics: The system has a Hamiltonian H. If the system is isolated, its state vector changes according to the Schrödinger equation:

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle \Longrightarrow |\psi(t)\rangle = \underbrace{e^{-iHt/\hbar}}_{\substack{\text{unitary evolution}\\ \text{operator } U(t)}}|\psi(0)\rangle$$

[Many people working in quantum info theory forget that unitary evolution comes from system Hamiltonians.]

Expectation values:
$$\langle A\rangle = \sum_\lambda \lambda p(\lambda)$$
$$= \langle\psi|\underbrace{\sum_\lambda \lambda P_\lambda}_{A}|\psi\rangle$$
$$= \langle\psi|A|\psi\rangle$$

All these will have to be generalized as we proceed.