Quantum information theory

Lectures 7-8

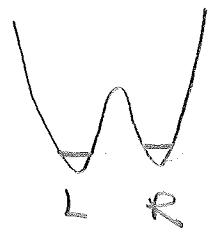
Qubits

Qubits: $D = 2$ 2-d Hilbert space

Orthonormal basis: $|0\rangle, |1\rangle \longleftarrow$ Fiducial basis, Computational basis, Standard basis

Examples:

1. Spin-$\frac{1}{2}$ particle: $|0\rangle = |\uparrow\rangle$, $|1\rangle = |\downarrow\rangle$
2. Photon polarization: $|0\rangle = |R\rangle$, $|1\rangle = |L\rangle$
3. Two-level atom: $|0\rangle = |e\rangle$, $|1\rangle = |g\rangle$
4. Double-well potential: $|0\rangle = |R\rangle$, $|1\rangle = |L\rangle$



Arbitrary pure state:

$$|\psi\rangle = a|0\rangle + b|1\rangle, \quad |a|^2 + |b|^2 = 1$$

$b = \langle 1|\psi\rangle = e^{i\phi}\sin(\theta/2), \quad 0 \le \phi \le 2\pi$ ← relative phase

a: choose real and positive by choice of overall phase:

$$a = \langle 0|\psi\rangle = \cos(\theta/2), \quad 0 \le \theta \le \pi$$

A measurement of $Z = |0\rangle\langle 0| - |1\rangle\langle 1|$ yields 1 ($|0\rangle$) with probability $|a|^2$ and -1 ($|1\rangle$) with probability $|b|^2$.

$$|\psi\rangle = \cos(\theta/2)|0\rangle + e^{i\phi}\sin(\theta/2)|1\rangle \equiv |\vec{n}\rangle = |+1, \vec{n}\rangle$$

$\vec{n}$: unit vector specified by θ and ϕ in polar coördinates

$-\vec{n}$: $c \leftrightarrow s$ $(\theta \leftrightarrow \pi - \theta)$

$\phi \rightarrow \phi + \pi$

$$|-\vec{n}\rangle = \sin(\theta/2)|0\rangle - e^{i\phi}\cos(\theta/2)|1\rangle = |-1; \vec{n}\rangle$$

$$\langle\vec{n}|-\vec{n}\rangle = 0$$

$|+1; \vec{e}_z\rangle = |\vec{e}_z\rangle = |0\rangle$

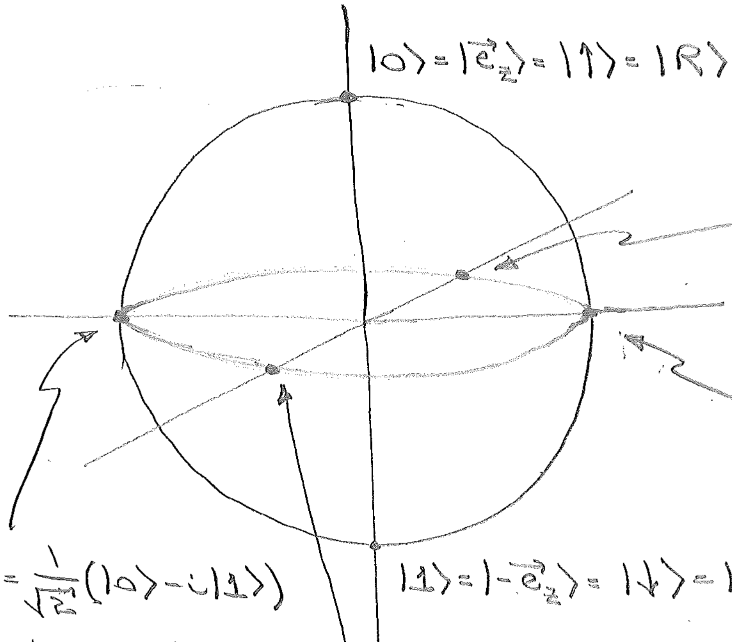
$|-1, \vec{e}_z\rangle = |-\vec{e}_z\rangle = |1\rangle$

$|x\rangle = |(-1)^x; \vec{e}_z\rangle$

Bloch sphere (spin-$\frac{1}{2}$)

Poincaré sphere (photon polarization)

(2)



$|-\vec{e}_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

spin in $-x$ direction

y polarization

$|\vec{e}_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

spin in y direction

polarization at 45° to x and y

$|-\vec{e}_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

spin in $-y$ direction

polarization at 45° to y and $-x$

$|\vec{n}\rangle$ — spin along $\vec{n}$, elliptical polarization

$|\vec{e}_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

spin in x direction

x polarization

$$P_{\vec{n}} = |\vec{n}\rangle\langle\vec{n}| = \overbrace{\cos^2(\theta/2)}^{\frac{1}{2}(1+\cos\theta)}|0\rangle\langle 0| + \overbrace{\sin^2(\theta/2)}^{\frac{1}{2}(1-\cos\theta)}|1\rangle\langle 1|$$
$$+ \underbrace{\cos(\theta/2)\sin(\theta/2)}_{\frac{1}{2}\sin\theta}\left(e^{-i\phi}|0\rangle\langle 1| + e^{i\phi}|1\rangle\langle 0|\right)$$

$$P_{\vec{n}} = \frac{1}{2}\Big(\underbrace{|0\rangle\langle 0| + |1\rangle\langle 1|}_{1} + \underbrace{\cos\theta}_{n_z}\underbrace{(|0\rangle\langle 0| - |1\rangle\langle 1|)}_{\equiv \sigma_z}$$
$$+ \underbrace{\sin\theta\cos\phi}_{n_x}\underbrace{(|0\rangle\langle 1| + |1\rangle\langle 0|)}_{\equiv \sigma_x}$$
$$+ \underbrace{\sin\theta\sin\phi}_{n_y}\underbrace{(-i|0\rangle\langle 1| + i|1\rangle\langle 0|)}_{\equiv \sigma_y}\Big)$$

$$P_{\vec{n}} = \frac{1}{2}\left(1 + \vec{n}\cdot\vec{\sigma}\right) \qquad \text{Spin-}\tfrac{1}{2}: \ \vec{S} = \tfrac{1}{2}\hbar\vec{\sigma}$$

Pauli matrices

$$\begin{cases} \sigma_x = \sigma_1 = X = |0\rangle\langle 1| + |1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_y = \sigma_2 = Y = -i|0\rangle\langle 1| + i|1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \sigma_z = \sigma_3 = Z = |0\rangle\langle 0| - |1\rangle\langle 1| \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{cases}$$

Properties of Pauli matrices:

① Hermitian: $\sigma_j = \sigma_j^\dagger$

Unitary: $\sigma_j^\dagger \sigma_j = \sigma_j^2 = 1$

② $\sigma_j \sigma_k = 1\,\delta_{jk} + i\epsilon_{jkl}\,\sigma_l \longleftarrow$ sum on repeated indices

$\uparrow$ antisymmetric symbol

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1 = i\sigma_3$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2 = i\sigma_1$$
$$\sigma_3\sigma_1 = -\sigma_1\sigma_3 = i\sigma_2$$

All products of Pauli matrices can be reduced to a single matrix

$$[\sigma_j, \sigma_k] = 2i\,\epsilon_{jkl}\,\sigma_l$$

$$[\sigma_1, \sigma_2] = 2i\sigma_3$$
$$[\sigma_2, \sigma_3] = 2i\sigma_1$$
$$[\sigma_3, \sigma_1] = 2i\sigma_2$$

$$[\sigma_j, \sigma_k]_+ = \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\,1$$

anticommutator

③ $\mathrm{tr}(\sigma_j) = 0 \implies \mathrm{tr}(\vec{n}\cdot\vec{\sigma}) = 0$

④ $\mathrm{tr}(\sigma_j^\dagger \sigma_k) = \mathrm{tr}(\sigma_j\sigma_k) = 2\delta_{jk} \longleftarrow$ orthogonality

(5) The operators $1, \sigma_1, \sigma_2, \sigma_3$ are a basis for the 4-d space operators (2×2 matrices), so any operator can be written as

$$A = A_0 1 + A_j \sigma_j = A_0 1 + \vec{A}\cdot\vec{\sigma} = A_\alpha \sigma_\alpha$$

$\sigma_0 \equiv 1$

$\alpha = 0,1,2,3$

$A^\dagger = A_\alpha^* \sigma_\alpha$

$A = A^\dagger \Rightarrow A_\alpha$ real

(6) Orthogonality: $\mathrm{tr}(\sigma_\alpha^\dagger \sigma_\beta) = \mathrm{tr}(\sigma_\alpha \sigma_\beta) = 2\delta_{\alpha\beta}$

$$A = A_\alpha \sigma_\alpha \Leftrightarrow A_\alpha = \tfrac{1}{2}\mathrm{tr}(\sigma_\alpha A)$$

(7) $A = A_\alpha \sigma_\alpha$ and $B = B_\alpha \sigma_\alpha$

$$\boxed{[A,B] = 2i\vec{A}\times\vec{B}\cdot\vec{\sigma}}$$

$[A, A^\dagger] = 2i\vec{A}\times\vec{A}^*\cdot\vec{\sigma}$

A normal $\Leftrightarrow \vec{A}\times\vec{A}^* = 0$

$\Leftrightarrow \vec{A} = e^{i\gamma}\underset{\substack{\uparrow \\ \text{real}}}{\vec{S}} = e^{i\gamma}|\vec{A}|\vec{m}$

$$AB = (A_0B_0 + \vec{A}\cdot\vec{B})1 + (A_0\vec{B} + B_0\vec{A} + i\vec{A}\times\vec{B})\cdot\vec{\sigma}$$

$$\Rightarrow \mathrm{tr}(AB) = 2(A_0B_0 + \vec{A}\cdot\vec{B}) = 2A_\alpha B_\alpha$$

Special case: $(\vec{n}\cdot\vec{\sigma})(\vec{m}\cdot\vec{\sigma}) = \vec{n}\cdot\vec{m}\,1 + i\vec{n}\times\vec{m}\cdot\vec{\sigma}$

(8) $P_{\vec{n}} = \frac{1}{2}(1 + \vec{n}\cdot\vec{\sigma})$

$$1 = P_{\vec{n}} + P_{-\vec{n}} = |\vec{n}\rangle\langle\vec{n}| + |-\vec{n}\rangle\langle-\vec{n}|$$

$$\vec{n}\cdot\vec{\sigma} = P_{\vec{n}} - P_{-\vec{n}} = \underbrace{|\vec{n}\rangle\langle\vec{n}| - |-\vec{n}\rangle\langle-\vec{n}|}_{\text{eigendecomposition of } \vec{n}\cdot\vec{\sigma}}$$

$\vec{n}\cdot\vec{\sigma}|\vec{n}\rangle = |\vec{n}\rangle$

$\vec{n}\cdot\vec{\sigma}|-\vec{n}\rangle = -|-\vec{n}\rangle$

(9) If $A = A_0 \mathbb{1} + \vec{A}\cdot\vec{\sigma}$ is Hermitian, define the unit vector $\vec{n} = \vec{A}/|\vec{A}|$, giving

$$A = A_0 \mathbb{1} + |\vec{A}|\,\vec{n}\cdot\vec{\sigma}$$

$$= (A_0 + |\vec{A}|)\,|\vec{n}\rangle\langle\vec{n}| + (A_0 - |\vec{A}|)\,|-\vec{n}\rangle\langle -\vec{n}|$$ eigen-decomposition

Normal operator: $\vec{A} = e^{i\chi}|\vec{A}|\vec{n}$; $(A_0 \pm e^{i\chi}|\vec{A}|)$ are eigenvalues

eigenvalues

eigenvectors

(10) Raising and lowering operators:

$$\sigma_\pm = \tfrac{1}{2}(\sigma_1 \pm i\sigma_2) \Longleftrightarrow \begin{array}{l} \sigma_1 = \sigma_+ + \sigma_- \\ \sigma_2 = -i(\sigma_+ - \sigma_-) \end{array}$$

$$\sigma_+ = |0\rangle\langle 1| \longleftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ raising operator

$$\sigma_- = |1\rangle\langle 0| \longleftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$ lowering operator

$$\sigma_+^\dagger = \sigma_-$$

$$\sigma_\pm^2 = 0 \qquad [\sigma_\pm, \sigma_\mp] = \pm\sigma_3$$

$$\sigma_\pm\sigma_\mp = \tfrac{1}{2}(\mathbb{1} \pm \sigma_3) \qquad [\sigma_\pm, \sigma_\mp]_+ = \mathbb{1}$$

$$\sigma_\pm\sigma_3 = \mp\sigma_\pm \qquad [\sigma_\pm, \sigma_3] = \mp 2\sigma_\pm$$

$$\sigma_3\sigma_\pm = \pm\sigma_\pm \qquad [\sigma_\pm, \sigma_3]_+ = 0$$

(10) $$e^{i\vec{n}\cdot\vec{\sigma}\gamma} = \mathbb{1}\cos\gamma + i\vec{n}\cdot\vec{\sigma}\sin\gamma$$

<u>Observables.</u> If the system is in the state $P_{\vec n} = |\vec n\rangle\langle\vec n|$, a measurement of $\vec m\cdot\vec\sigma = |\vec m\rangle\langle\vec m| - |-\vec m\rangle\langle-\vec m|$ yields $+1$ ($|\vec m\rangle$) with probability

$$|\langle\vec m|\vec n\rangle|^2 = \mathrm{tr}(P_{\vec m}P_{\vec n}) = \tfrac{1}{2}(1+\vec m\cdot\vec n)$$

and -1 ($|-\vec m\rangle$) with probability

$$|\langle-\vec m|\vec n\rangle|^2 = \mathrm{tr}(P_{-\vec m}P_{\vec n}) = \tfrac{1}{2}(1-\vec m\cdot\vec n)$$

<u>Unitary dynamics.</u> Any unitary operator can be written as

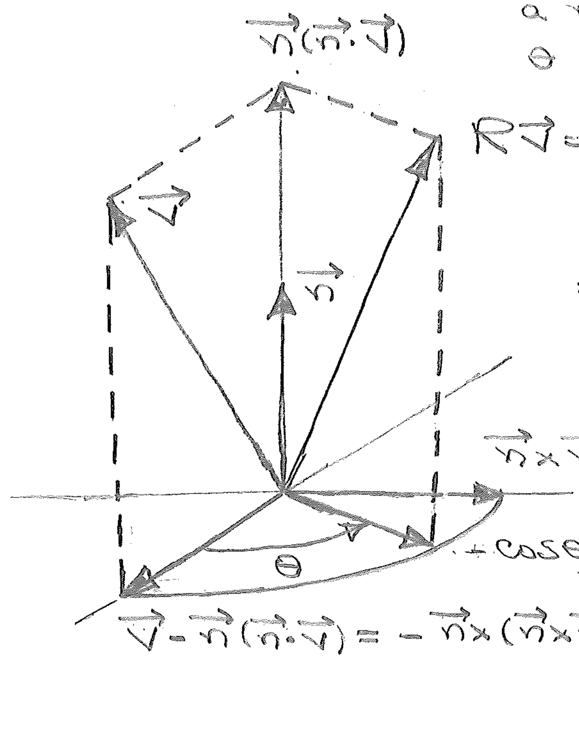
$$U = e^{i(\delta 1 - \vec n\cdot\vec\sigma\,\theta/2)} = \underbrace{e^{i\delta}}_{\text{global phase}}\;\underbrace{e^{-i\vec n\cdot\vec\sigma\,\theta/2}}_{U_R = \exp\left(-\frac{i}{\hbar}\vec n\cdot\vec S\,\theta\right)}$$

spin

canonical form of rotation operator

<u>Key fact:</u> $U_R^\dagger\vec\sigma U_R = \underbrace{R_{\vec n}(\theta)}\vec\sigma$

orthogonal matrix for a rotation by angle θ about $\vec n$ $\longrightarrow R^TR = I$



$$R\vec V = \vec n(\vec n\cdot\vec V) - \cos\theta\,\vec n\times(\vec n\times\vec V) + \sin\theta\,\vec n\times\vec V$$

$$= +\vec V\cos\theta + (1-\cos\theta)\vec n(\vec n\cdot\vec V) + \sin\theta\,\vec n\times\vec V$$

$U_R^\dagger\sigma_j U_R = \hat e_j\cdot R\vec\sigma = \hat e_j\cdot R\hat e_k\sigma_k = \underbrace{(\hat e_j\cdot R\hat e_k)}_{R_{jk}}\sigma_k = R_{jk}\sigma_k \Rightarrow U_R^\dagger\sigma_j U_R = R_{jk}\sigma_k$, $U_R\sigma_j U_R^\dagger = \sigma_k R_{kj}$

Consequence: $U_R \vec{\sigma}\cdot\vec{m}\, U_R^\dagger = R^T\vec{\sigma}\cdot\vec{m} = \vec{\sigma}\cdot R\vec{m}$

$$(\vec{\sigma}\cdot R\vec{m})U_R|\vec{m}\rangle = U_R \underbrace{\vec{\sigma}\cdot\vec{m}|\vec{m}\rangle}_{+|\vec{m}\rangle} = U_R|\vec{m}\rangle$$

$$\Rightarrow U_R|\vec{m}\rangle = e^{i\phi(R,\vec{m})}|R\vec{m}\rangle$$

Unitary dynamics rotates states on the Bloch sphere.

<u>Example:</u> Hamiltonian (spin-1/2) $H = -\vec{m}\cdot\vec{B} = \frac{1}{2}\gamma B\hbar Z$

↑ magnetic moment; magnetic field

Choose $\vec{B} = B\vec{e}_z$

$\vec{m} = -\gamma\vec{S} = -\frac{1}{2}\gamma\hbar\vec{\sigma}$

$-\vec{m}\cdot\vec{B} = \frac{1}{2}\gamma B\hbar Z$

Unitary evolution operator:

$$U(t) = e^{-iHt/\hbar} = e^{-i(\gamma Bt)/2)Z}$$

rotation about $\vec{e}_z$ by angle γBt

Comment: Why put $|0\rangle$ at the top of the Bloch sphere ($\sigma_z|0\rangle = |0\rangle$), when $|0\rangle$ suggests the ground state, which conventionally goes at the bottom?

$$\sigma_z |x\rangle = (-1)^x |x\rangle \longleftarrow$$

x: 0 or 1

State at top has eigenvalue +1, whereas state at bottom has eigenvalue −1.

This is all because of the two ways of doing mod-2 arithmetic — multiplicative ($(-1)^x$) or additive (x).

Tricky point:

$$U_R = e^{-i\vec{n}\cdot\vec{\sigma}\theta/2} = e^{-i\theta/2}|\vec{n}\rangle\langle\vec{n}| + e^{+i\theta/2}|-\vec{n}\rangle\langle-\vec{n}|$$

$$\Rightarrow \det U_R = e^{-i\theta/2}e^{+i\theta/2} = 1$$

$$\det U = e^{2i\delta}$$

$$U = e^{i\delta}U_R$$

Requiring U to have unit determinant, i.e., to be in $SU(2)$, means that $\delta = 0$ or π. But we don't need the π case, which changes the sign of U, because advancing θ by 2π also changes the sign of U (double-valued representation of $SO(3)$).