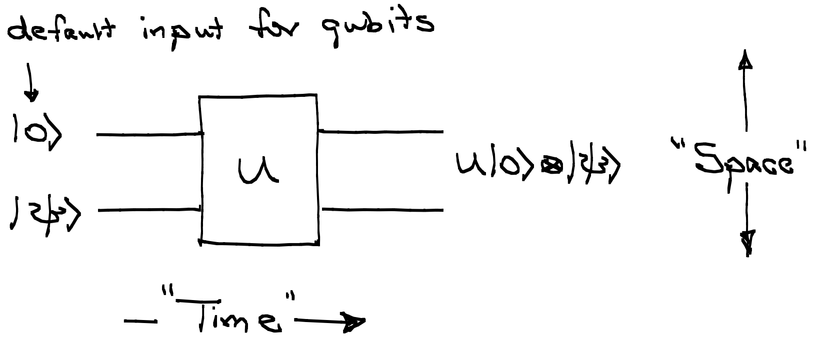
Quantum information theory

Lectures 18-20

Quantum circuit model

Quantum circuit model

Quantum states propagate along "wires" through "gates"



Qubit example

The power of quantum circuit diagrams is that they clearly display temporal and spatial relations that are hard to convey and not readily apparent in algebraic expressions.

Classical gates

IDENTITY	$a \longrightarrow a$	✓	
NOT	$a \longrightarrow \bar{a} = 1 \oplus a$	✓	
FANOUT	$a \longrightarrow a, a, a$	×	no cloning
CROSSOVER (SWAP)	$a, b \longrightarrow b, a$	✓	
AND	$a, b \longrightarrow ab$	×	
OR	$a, b \longrightarrow \overline{\bar{a}\bar{b}} = 1 \oplus (1\oplus a)(1\oplus b) = a \oplus b \oplus ab$	×	
XOR	$a, b \longrightarrow a \oplus b$	×	
NAND	$a, b \longrightarrow \overline{ab} = 1 \oplus ab$	×	
NOR	$a, b \longrightarrow \bar{a}\bar{b} = (1\oplus a)(1\oplus b) = 1 \oplus a \oplus b \oplus ab$	×	

AND, OR, XOR, NAND, NOR → irreversible

quantum gates must be reversible, with same number of inputs and outputs

Truth table

XOR

a	b	$a\oplus b$
0	0	0
0	1	1
1	0	1
1	1	0

AND

a	b	ab
0	0	0
0	1	0
1	0	0
1	1	1

Important single-qubit gates:

$ie^{-iX\pi/2} = X \longleftrightarrow \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\quad |0\rangle \rightarrow |1\rangle$, $|1\rangle \rightarrow |0\rangle$ $\quad |a\rangle \rightarrow |\bar{a}\rangle = |1\oplus a\rangle$ NOT

$ie^{-iZ\pi/2} = Z \longleftrightarrow \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ $\quad |0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow -|1\rangle$ $\quad |a\rangle \longrightarrow (-1)^a|a\rangle$ SIGN

$-e^{-iY\pi/2} = iY = ZX \longleftrightarrow \begin{pmatrix}0&1\\-1&0\end{pmatrix}$ $\quad |0\rangle \rightarrow -|1\rangle$, $|1\rangle \rightarrow |0\rangle$ $\quad |a\rangle \longrightarrow (-1)^{\bar{a}}|\bar{a}\rangle = -(-1)^a|1\oplus a\rangle$

$ie^{-i\frac{1}{\sqrt{2}}(X+Y)\pi/2} = \frac{1}{\sqrt{2}}(X+Z) \equiv H \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}$ $\quad |0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$, $|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)$ $\quad |a\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle+(-1)^a|1\rangle)$

HADAMARD

↑

all 180° rotations, with phases chosen so that the operator is both unitary and Hermitian and thus squares to I.

$$H = H^\dagger \Rightarrow H^2 = I$$

$$HXH = Z$$
$$HZH = X$$
$$HYH = Y$$

cf. 90° rotation about y:

$$U = e^{-iY\pi/4} = \frac{1}{\sqrt{2}}(I - iY) = \frac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$$

$$UXU^\dagger = -Z$$
$$UYU^\dagger = Y$$
$$UZU^\dagger = X$$

$e^{i\pi/4}e^{-iZ\pi/4} \equiv S \longleftrightarrow \begin{pmatrix}1&0\\0&i\end{pmatrix}$ $\quad |0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow i|1\rangle$ $\quad |a\rangle \rightarrow i^a|a\rangle$ PHASE

↑

90° rotation about z $\qquad S^2 = Z$

$$SXS^\dagger = Y$$
$$SYS^\dagger = -X$$
$$SZS^\dagger = Z$$

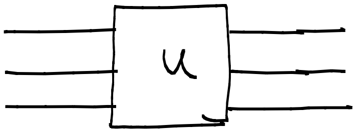
$e^{i\pi/8}e^{-iZ\pi/8} \equiv T \longleftrightarrow \begin{pmatrix}1&0\\0&e^{i\pi/4}\end{pmatrix}$ $\quad |0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow e^{i\pi/4}|1\rangle$ $\quad |a\rangle \longrightarrow e^{ia\pi/4}$ T

↑

45° rotation about z $\qquad T^2 = S$

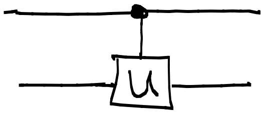
$$TXT^\dagger = \frac{1}{\sqrt{2}}(X+Y)$$
$$TYT^\dagger = \frac{1}{\sqrt{2}}(-X+Y)$$
$$TZT^\dagger = Z$$

Multiple qubit gates:



too general

Controlled-unitaries:



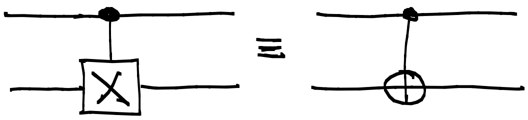
$$P_0 \otimes I + P_1 \otimes U \longleftrightarrow \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 1 & 0 & 0 \\ 10 & 0 & 0 & \multicolumn{2}{c}{\boxed{U}} \\ 11 & 0 & 0 & & \end{array}$$

Truth table? U must be a permutation.

$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |1\rangle \otimes U|0\rangle$$
$$|11\rangle \longrightarrow |1\rangle \otimes U|1\rangle$$

$$|a,b\rangle \rightarrow |a\rangle \otimes U^a|b\rangle$$

CNOT:



$$P_0 \otimes I + P_1 \otimes X \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\underbrace{\boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}}_{X}$}} \\ 0 & 0 & & \end{pmatrix}$$

$$(CNOT)^\dagger = CNOT$$
$$\Longrightarrow \quad (CNOT)^2 = I$$

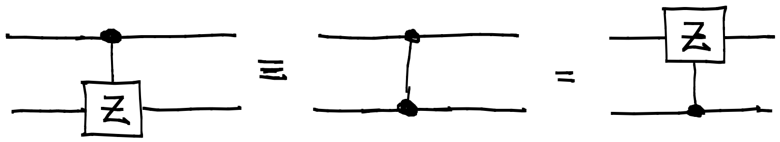
$$|00\rangle \longrightarrow |00\rangle$$
$$|01\rangle \longrightarrow |01\rangle$$
$$|10\rangle \longrightarrow |11\rangle$$
$$|11\rangle \longrightarrow |10\rangle$$

$$|a,b\rangle \longrightarrow |a, a\oplus b\rangle$$

This is a classical truth table.

XOR of inputs placed in second qubit

CSIGN:



sometimes called
CPHASE

$$P_0\otimes I + P_1\otimes Z \longleftrightarrow \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&\underbrace{\boxed{\begin{matrix}1&0\\0&-1\end{matrix}}}_{\hookrightarrow Z}\end{pmatrix}$$

$$(CSIGN)^{\dagger} = CSIGN$$

$$\Rightarrow (CSIGN)^2 = I$$

$|00\rangle \longrightarrow |00\rangle$

$|01\rangle \longrightarrow |01\rangle$

$|10\rangle \longrightarrow |10\rangle$

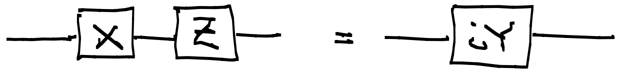
$|11\rangle \longrightarrow -|11\rangle$

$$|a,b\rangle \longrightarrow \underset{\uparrow}{(-1)^{ab}}\,|a,b\rangle$$

phase gives AND
of inputs

Circuit identities

①

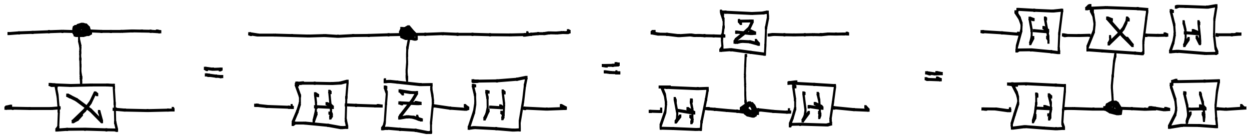


order

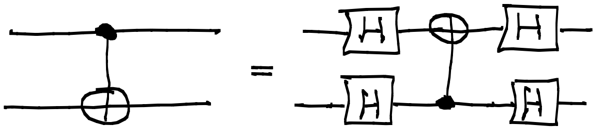
②

HADAMARD swaps standard basis, $|0\rangle, |1\rangle$, and the X basis, $|\pm\rangle \equiv H{|0\rangle \atop |1\rangle} = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$.

③



i.e.,



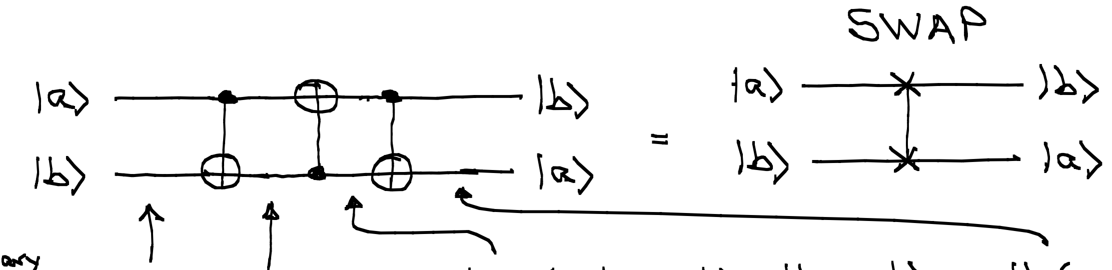
cf.

$$P_0\otimes I + P_1\otimes X$$

$$= P_0\otimes(P_x + P_{-x}) + P_1\otimes(P_x - P_{-x})$$

$$= I\otimes P_x + Z\otimes P_{-x}$$

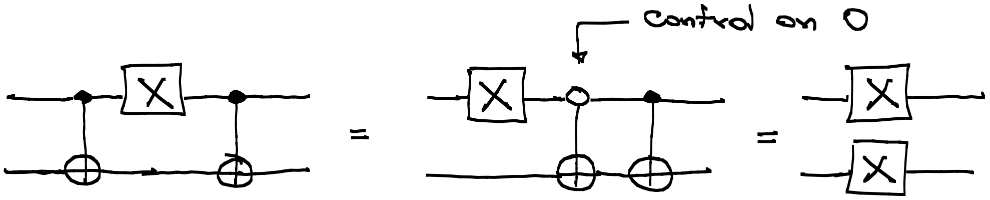
④



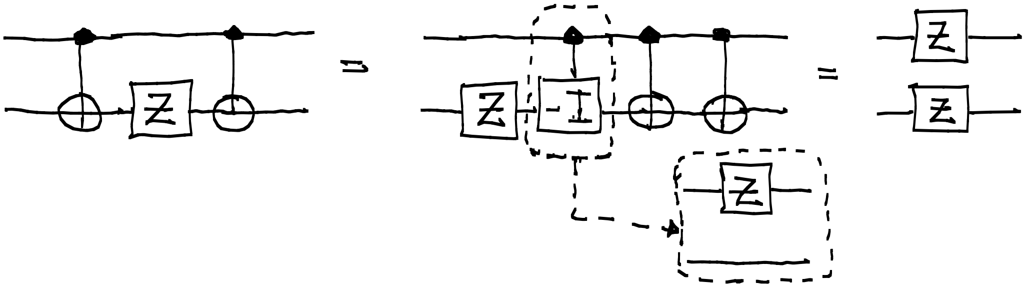
using binary
arithmetic

$$|a,b\rangle \to |a, a\oplus b\rangle \longrightarrow |a\oplus(a\oplus b), a\oplus b\rangle = |b, a\oplus b\rangle \longrightarrow |b, (a\oplus b)\oplus b\rangle = |b,a\rangle$$

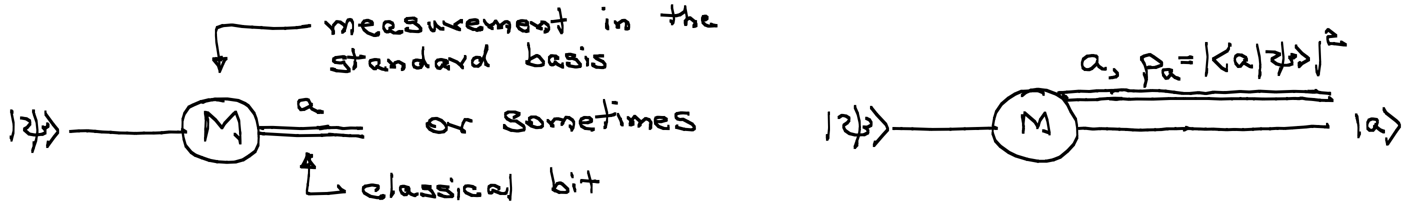
⑤



⑥

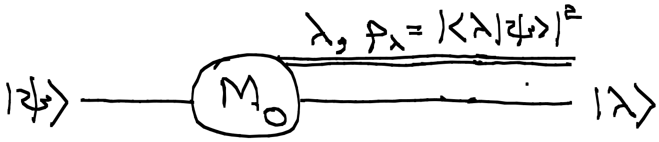


Measurements

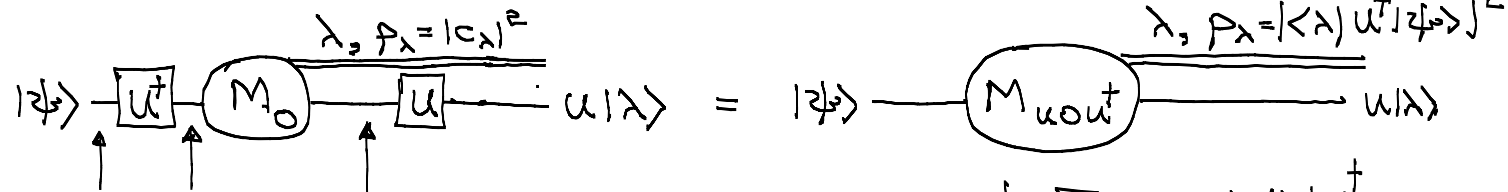


Two tricks:

① Measurement in another basis:



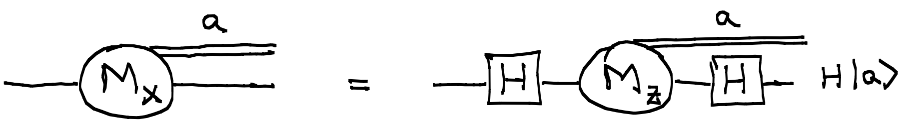
$$O = \sum_\lambda \lambda |\lambda\rangle\langle\lambda|$$



$|\psi\rangle = \sum_\lambda c_\lambda U|\lambda\rangle \quad \sum_\lambda c_\lambda |\lambda\rangle \quad |\lambda\rangle$

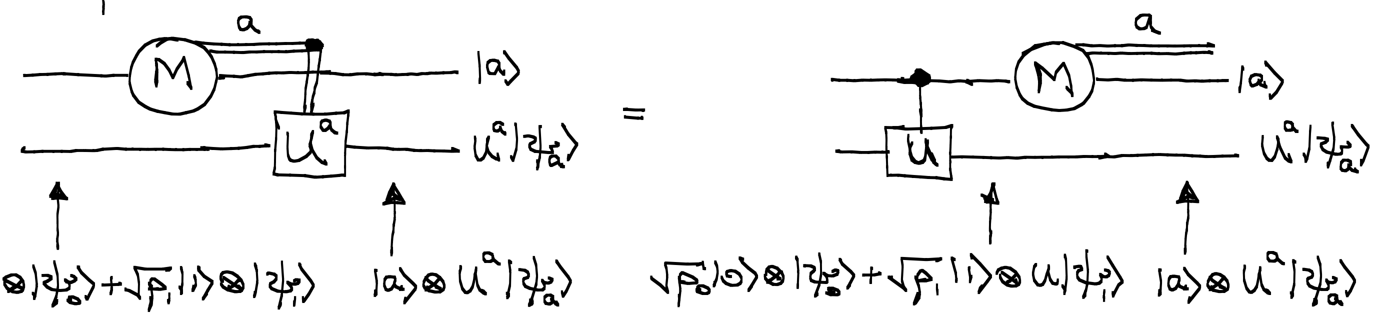
$$UOU^\dagger = \sum_\lambda \lambda\, U|\lambda\rangle\langle\lambda|U^\dagger$$

Example:



Works also for measuring a degenerate observable.

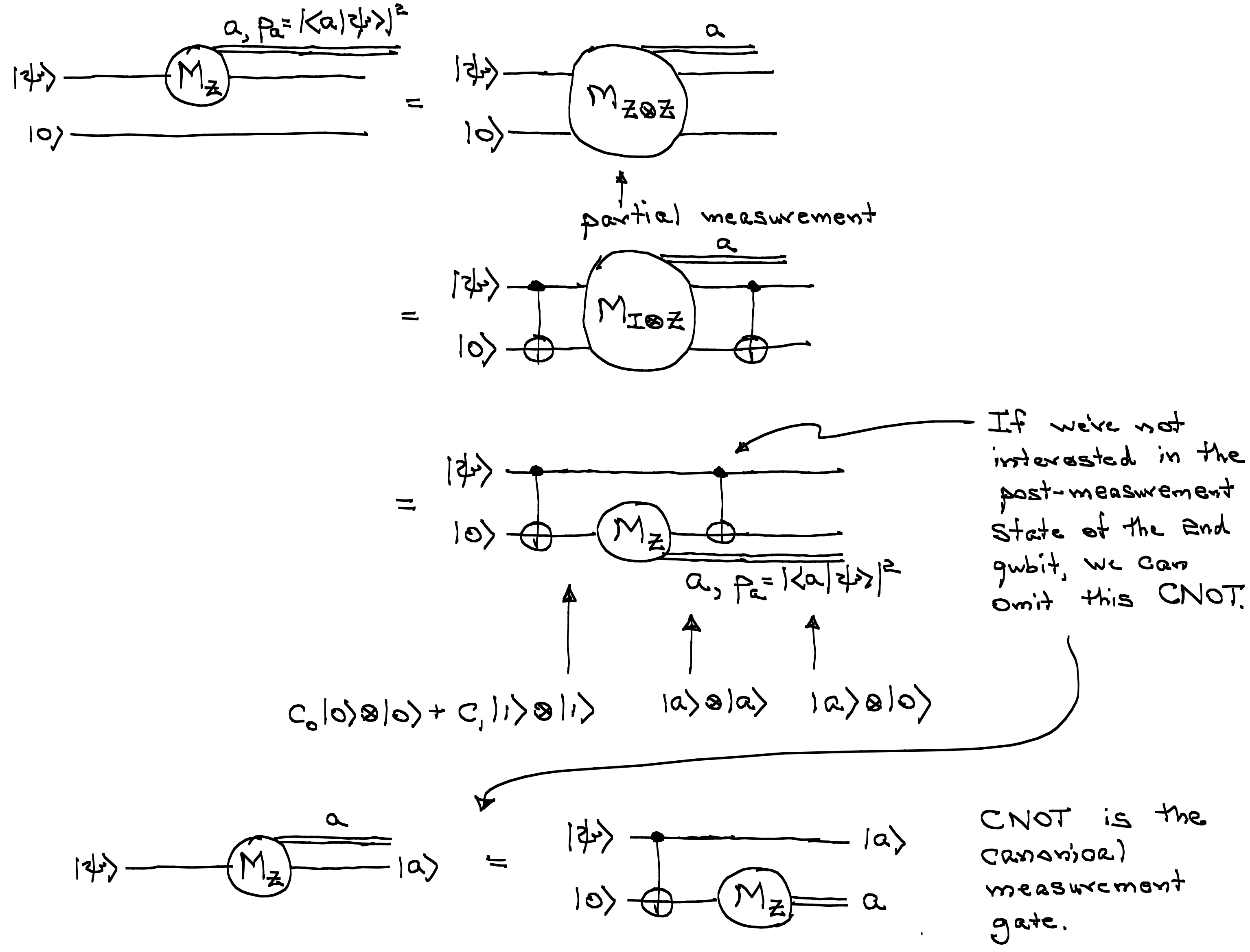
② Principle of deferred measurement



$\sqrt{p_0}|0\rangle\otimes|\psi_0\rangle+\sqrt{p_1}|1\rangle\otimes|\psi_1\rangle$ $\quad |a\rangle\otimes U^a|\psi_a\rangle$ $\quad \sqrt{p_0}|0\rangle\otimes|\psi_0\rangle+\sqrt{p_1}|1\rangle\otimes U_1|\psi_1\rangle$ $\quad |a\rangle\otimes U^a|\psi_a\rangle$

relative-state decomposition

Measurement models



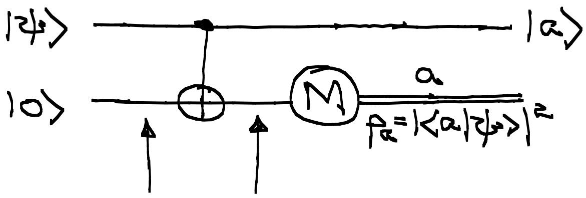
$c_0|0\rangle\otimes|0\rangle+c_1|1\rangle\otimes|1\rangle$ $\quad |a\rangle\otimes|a\rangle$ $\quad |a\rangle\otimes|0\rangle$

If we're not interested in the post-measurement state of the 2nd qubit, we can omit this CNOT.

CNOT is the canonical measurement gate.

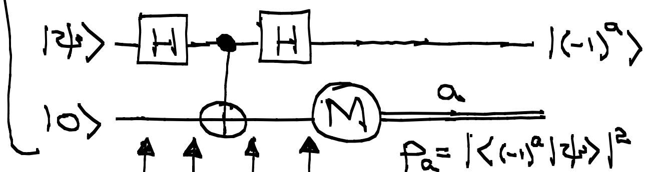
General measurement models can be developed from a Neumark extension, but it's often useful just to guess an answer. Here's an example.

A 4-outcome POVM:



$|\psi\rangle\otimes|0\rangle$ $\quad$ $\langle 0|\psi\rangle|00\rangle + \langle 1|\psi\rangle|11\rangle$

OR



$|\psi\rangle\otimes|0\rangle$ $\quad$ $\langle +|\psi\rangle|00\rangle + \langle -|\psi\rangle|11\rangle$

$(|0\rangle\langle +|\psi\rangle + |1\rangle\langle -|\psi\rangle)\otimes|0\rangle$

$\langle +|\psi\rangle|+\rangle\otimes|0\rangle + \langle -|\psi\rangle|-\rangle\otimes|1\rangle$

$$|\psi\rangle = |0\rangle\langle 0|\psi\rangle + |1\rangle\langle 1|\psi\rangle$$
$$= |+\rangle\langle +|\psi\rangle + |-\rangle\langle -|\psi\rangle$$

$$|\pm\rangle = |\pm\vec{e}_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

$H|0\rangle = |+\rangle \qquad H|+\rangle = |0\rangle$

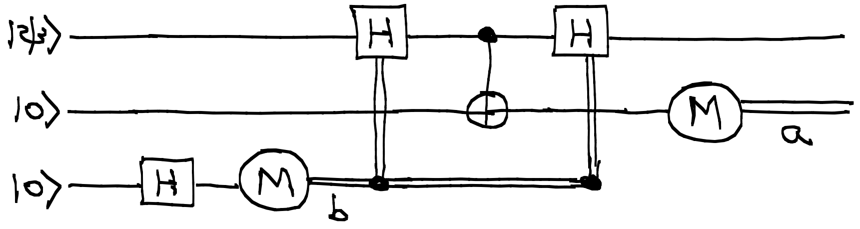
$H|1\rangle = |-\rangle \qquad H|-\rangle = |1\rangle$

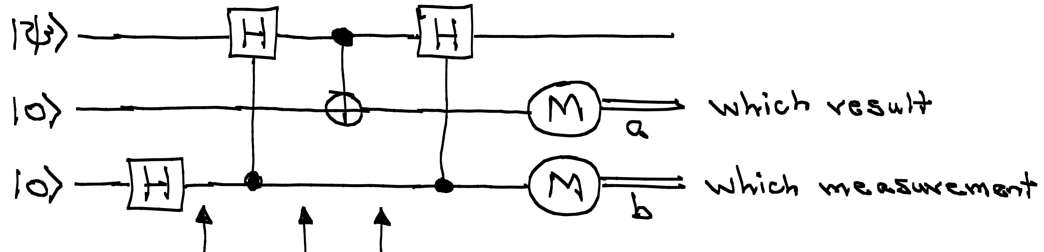
$$H|\psi\rangle = |0\rangle\langle +|\psi\rangle + |1\rangle\langle -|\psi\rangle$$

Kraus operators	POVM elements
$A_{00} = \frac{1}{\sqrt{2}}\|0\rangle\langle 0\|$	$E_{00} = \frac{1}{2}\|0\rangle\langle 0\|$
$A_{10} = \frac{1}{\sqrt{2}}\|1\rangle\langle 1\|$	$E_{10} = \frac{1}{2}\|1\rangle\langle 1\|$
$A_{01} = \frac{1}{\sqrt{2}}\|+\rangle\langle +\|$	$E_{01} = \frac{1}{2}\|+\rangle\langle +\|$
$A_{11} = \frac{1}{\sqrt{2}}\|-\rangle\langle -\|$	$E_{11} = \frac{1}{2}\|-\rangle\langle -\|$

A_{ab}
which measurement
which result

Use a 3rd qubit to flip the coin.





$|\psi\rangle\otimes|0\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$\frac{1}{\sqrt{2}}\left(|\psi\rangle\otimes|00\rangle + H|\psi\rangle\otimes|01\rangle\right)$

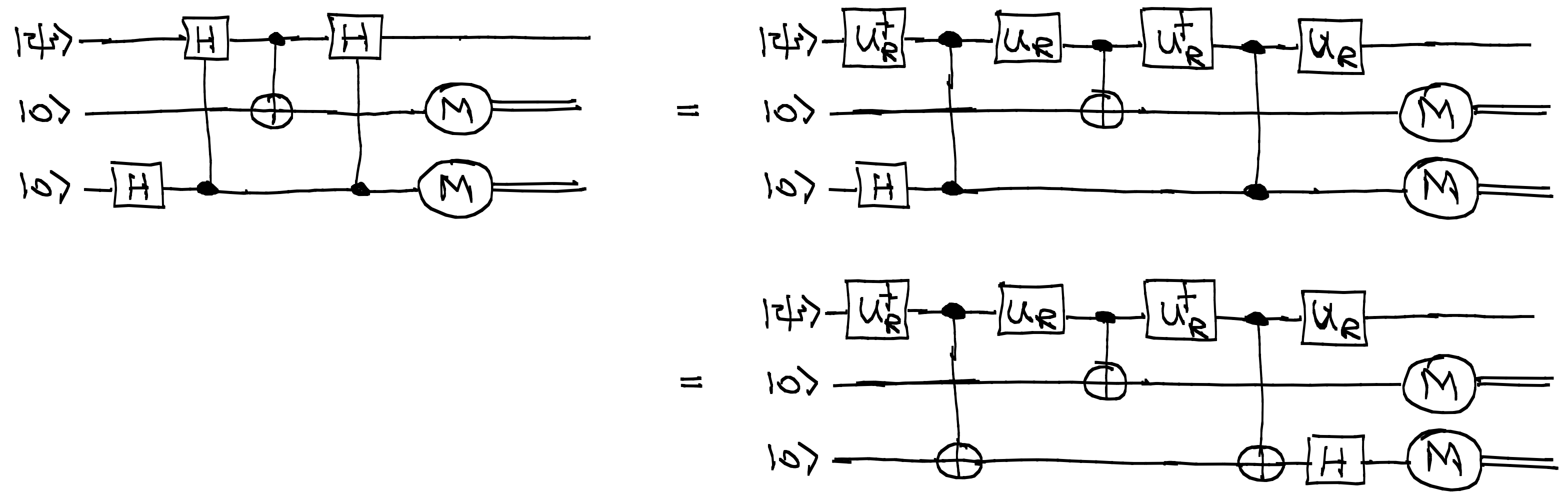
$\frac{1}{\sqrt{2}}\left(\langle 0|\psi\rangle|000\rangle + \langle 1|\psi\rangle|110\rangle\right) + \frac{1}{\sqrt{2}}\left(\langle +|\psi\rangle|001\rangle + \langle -|\psi\rangle|111\rangle\right)$

$\frac{1}{\sqrt{2}}\left(\langle 0|\psi\rangle|0\rangle\otimes|00\rangle + \langle 1|\psi\rangle|1\rangle\otimes|10\rangle\right) + \frac{1}{\sqrt{2}}\left(\langle +|\psi\rangle|+\rangle\otimes|01\rangle + \langle -|\psi\rangle|-\rangle\otimes|11\rangle\right)$

We can "improve" this circuit a little, so as to target all the controlled operations (as CNOTs) onto the measured qubits. First, let R be a rotation that satisfies $R\vec{e}_z = \frac{1}{\sqrt{2}}(\vec{e}_x+\vec{e}_z)$, e.g., a 45° rotation about y or a 180° rotation about $\vec{e}_z\cos\frac{\pi}{8}+\vec{e}_x\sin\frac{\pi}{8}$. Then

$$\begin{array}{l} \vec{\sigma}\cdot R\vec{e}_z = \frac{1}{\sqrt{2}}(X+Z) = H \\ \quad \| \\ R^T\vec{\sigma}\cdot\vec{e}_z = U_R\vec{\sigma}U_R^\dagger = U_R Z U_R^\dagger \end{array}$$

Now transform the measuring circuit in the following ways:



This form is by no means obvious without the use of circuit diagrams.

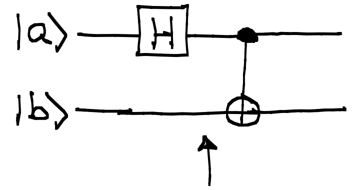
<u>Making Bell states</u>: $|\beta_{ab}\rangle = Z^aX^b\otimes I|\beta_{00}\rangle = I\otimes X^bZ^a|\beta_{00}\rangle$

phase bit

$$= \frac{1}{\sqrt{2}}(|0b\rangle + (-1)^a|1\bar{b}\rangle)$$

parity bit

$$= \mathrm{CNOT}(H\otimes I)|a,b\rangle$$

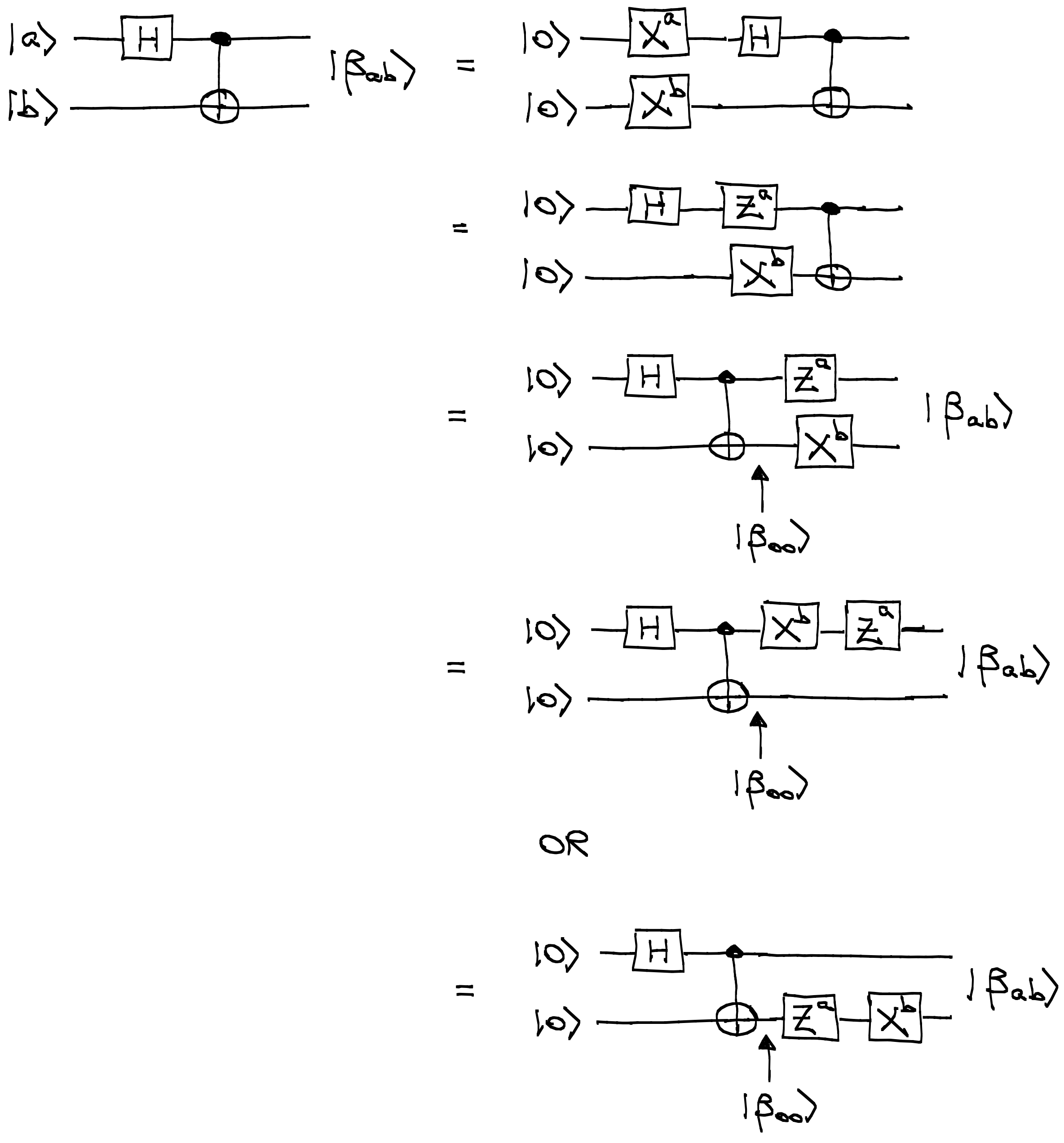


$$\frac{1}{\sqrt{2}}(|0b\rangle + (-1)^a|1\bar{b}\rangle) = |\beta_{ab}\rangle$$

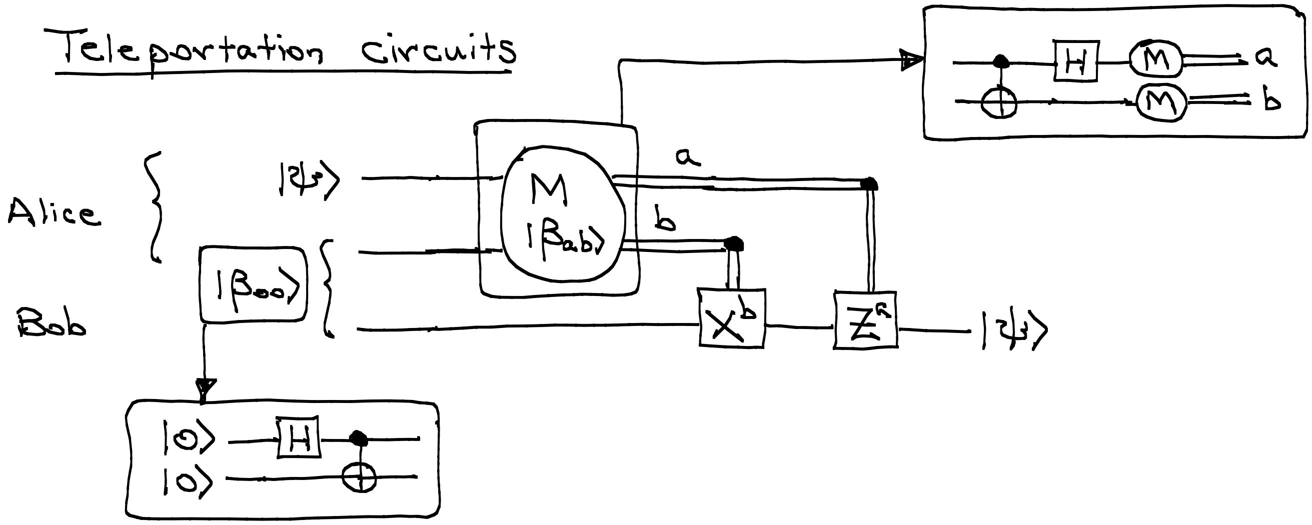
$$\frac{1}{\sqrt{2}}(|0\rangle + (-1)^a|1\rangle)\otimes|b\rangle$$

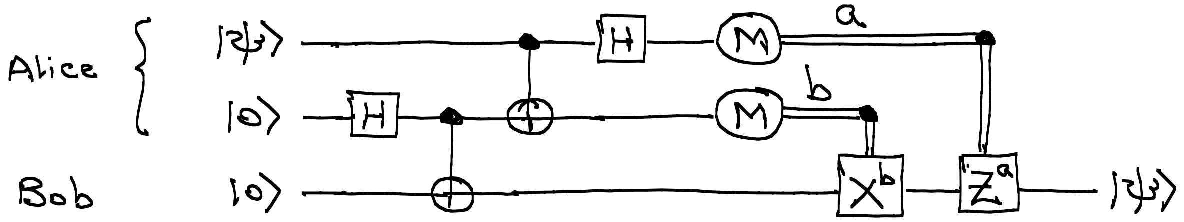
$$\boxed{\begin{aligned} &A\otimes I\frac{1}{\sqrt{D}}\sum_k|e_k\rangle\otimes|f_k\rangle \\ &= \frac{1}{\sqrt{D}}\sum_{j,k}|e_j\rangle\langle e_j|A|e_k\rangle\otimes|f_k\rangle \\ &= \frac{1}{\sqrt{D}}\sum_{j,k}|e_j\rangle\otimes|f_k\rangle\langle e_k|A^T|e_j\rangle \\ &= I\otimes A^T\frac{1}{\sqrt{D}}\sum_j|e_j\rangle\otimes|f_j\rangle \end{aligned}}$$

Relation between two expressions for Bell states:



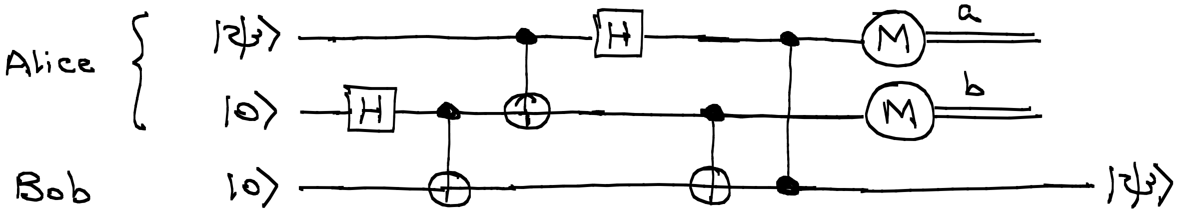
Teleportation circuits





Teleportation circuit

We can refine this circuit in one other way, by using the principle of deferred measurement to move the final controls through the measurement.

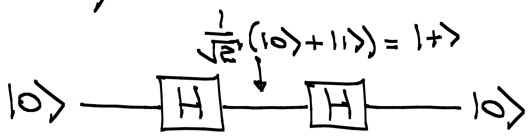


Reversible teleportation circuit

The measurements become irrelevant and can be deleted, leaving a reversible teleportation circuit. The measurements tell us after the fact what action was taken by the CNOT and CSIGN gates. Of course, one of the points of teleportation is that Alice and Bob interact only via classical communication (2 bits from Alice to Bob), whereas the reversible circuit has a direct quantum interaction.

Quantum eraser

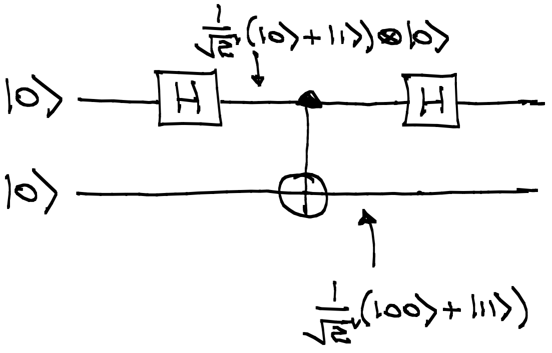
Ramsey interferometer



$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$

$H|0\rangle = |+\rangle$ $\quad H|+\rangle = |0\rangle$

$H|1\rangle = |-\rangle$ $\quad H|-\rangle = |1\rangle$

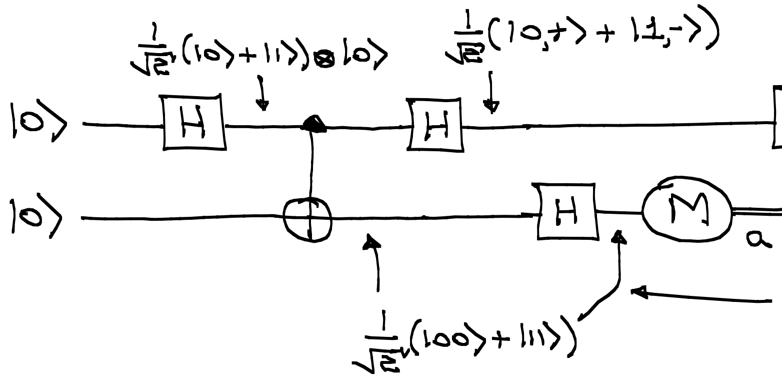


$$\rho = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$\frac{1}{\sqrt{2}}(|+,0\rangle + |-,1\rangle) = \frac{1}{\sqrt{2}}(|0,+\rangle + |1,-\rangle)$$

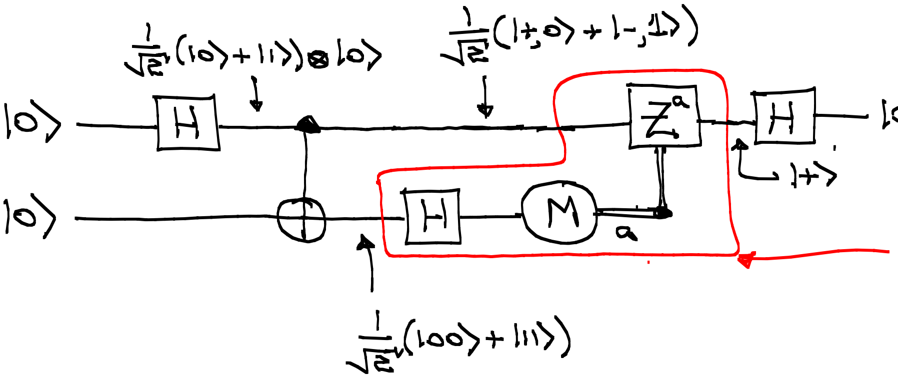
The entanglement produced by the C-NOT (potential measurement) destroys the interference

Restoring the interference

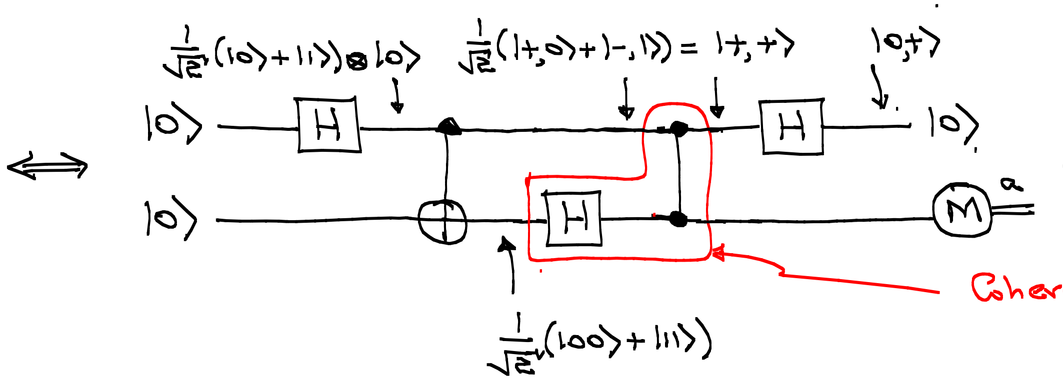


This follows from

$$H\otimes H|\beta_{00}\rangle = I\otimes \underbrace{H^T H}_{I}|\beta_{00}\rangle = |\beta_{00}\rangle$$

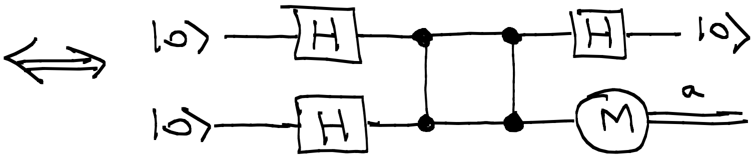


Quantum eraser: a measurement-based way to remove the entanglement



Measurement is irrelevant

Coherent disentangler



This makes it clear that the disentangler works.

Non-rank-one POVM

Consider the qubit POVM

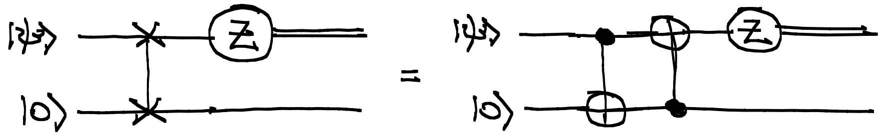
$$E_0 = \frac{1}{3}|1\rangle\langle 1| + |0\rangle\langle 0| = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}I$$

$$E_1 = \frac{2}{3}|1\rangle\langle 1|$$

This POVM is what comes from mixing two outcomes of the trine, but with things relabeled a bit.

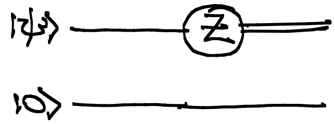
This measurement can be gotten by flipping a coin that has $\frac{1}{3}$ probability of coming up 0, after which one performs a measurement that always gives result 0, and $\frac{2}{3}$ probability of coming up 1, after which one does a Z measurement.

probability $\frac{1}{3}$

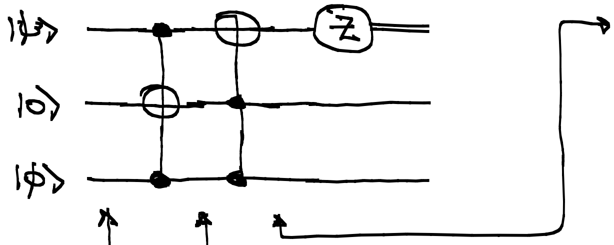


OR

probability $\frac{2}{3}$



Put in another qubit to do the flipping:



$$\sqrt{\frac{2}{3}}a|000\rangle + \sqrt{\frac{2}{3}}b|100\rangle + \sqrt{\frac{1}{3}}a|001\rangle + \sqrt{\frac{1}{3}}b|011\rangle$$

$$= \sqrt{\frac{2}{3}}|\psi\rangle\otimes|00\rangle + \sqrt{\frac{1}{3}}a|0\rangle\otimes|01\rangle + \sqrt{\frac{1}{3}}b|0\rangle\otimes|11\rangle$$

$$\rho = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{3}|0\rangle\langle 0|$$

$$\underbrace{(a|0\rangle + b|1\rangle)}_{|\psi\rangle}\otimes|0\rangle\otimes\underbrace{\left(\sqrt{\frac{2}{3}}|0\rangle + \sqrt{\frac{1}{3}}|1\rangle\right)}_{|\phi\rangle} = \sqrt{\frac{2}{3}}a|000\rangle + \sqrt{\frac{2}{3}}b|100\rangle + \sqrt{\frac{1}{3}}a|001\rangle + \sqrt{\frac{1}{3}}b|101\rangle$$

$$\sqrt{\frac{2}{3}}a|000\rangle + \sqrt{\frac{2}{3}}b|100\rangle + \sqrt{\frac{1}{3}}a|001\rangle + \sqrt{\frac{1}{3}}b|111\rangle$$

Marginal density operator of top qubit, $\rho = \frac{2}{3}|\psi\rangle\langle\psi| + \frac{1}{3}|0\rangle\langle 0|$, gives probabilities

$$p_0 = \langle 0|\rho|0\rangle = \frac{2}{3}|a|^2 + \frac{1}{3}, \quad p_1 = \langle 1|\rho|1\rangle = \frac{2}{3}|b|^2,$$

which are those of the desired POVM. To make a measurement model, we simply model the Z measurement: