

Quantum information theory

Lectures 4-5

Linear algebra and axioms of quantum mechanics

The behavior of quantum systems is described in a complex vector space with an inner product.

$$|\psi\rangle = \sum_{j=1}^D c_j |\psi_j\rangle$$

↑ dimension of the space  
↑ basis of linearly independent vectors  
↑ vector "ket"

A complex inner-product vector space is called a Hilbert Space.

Inner product:  $(|\phi\rangle, |\psi\rangle) = \langle\phi|\psi\rangle$  ← definition of "bra"  $\langle\phi|$  as a linear function from kets to  $\mathbb{C}$ .

① Linear in  $|\psi\rangle$ :  $(|\phi\rangle, a|\psi\rangle + b|\chi\rangle) = a(|\phi\rangle, |\psi\rangle) + b(|\phi\rangle, |\chi\rangle)$

② Complex symmetric:  $(|\phi\rangle, |\psi\rangle) = (|\psi\rangle, |\phi\rangle)^*$  ← Complex conjugate

⇒ antilinearity in  $|\phi\rangle$ :  $(a|\phi\rangle + b|\chi\rangle, |\psi\rangle) = a^*(|\phi\rangle, |\psi\rangle) + b^*(|\chi\rangle, |\psi\rangle)$

③  $(|\psi\rangle, |\psi\rangle) \geq 0$ , with equality iff  $|\psi\rangle = 0$

$$|\phi\rangle \longleftrightarrow \langle\phi|$$

$$a|\phi\rangle + b|\chi\rangle \longleftrightarrow a^*\langle\phi| + b^*\langle\chi|$$

Often write  $|e_j\rangle = \underline{e_j}$

Orthonormal basis  $|e_j\rangle, j=1, \dots, D$ :  $\langle e_j | e_k \rangle = \delta_{jk}$

$$|\psi\rangle = \sum_j c_j |e_j\rangle = \sum_j |e_j\rangle \langle e_j | \psi \rangle$$

↓  
=  $\langle e_j | \psi \rangle$

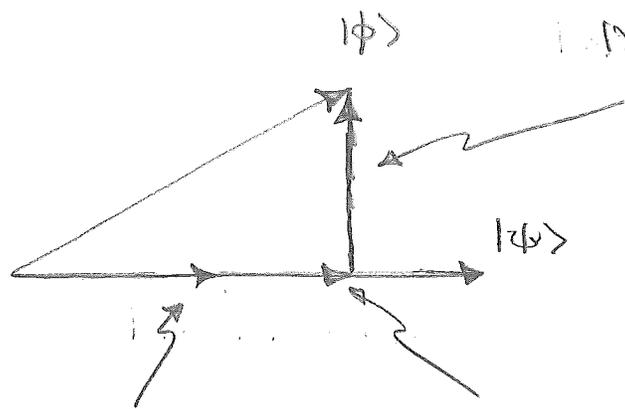
$$|\phi\rangle = \sum_j d_j |e_j\rangle = \sum_j |e_j\rangle \langle e_j | \phi \rangle$$

$$\langle\phi|\psi\rangle = \sum_j d_j^* c_j = \sum_j \langle\phi|e_j\rangle \langle e_j|\psi\rangle$$

$|\phi\rangle \leftrightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$  representation  
 $\langle\phi| = \sum_j d_j^* \langle e_j|$   
 $\leftrightarrow (d_1^*, \dots, d_D^*)$   
 $\langle\phi|\psi\rangle = (d_1^* \dots d_D^*) \begin{pmatrix} c_1 \\ \vdots \\ c_D \end{pmatrix}$

Schwarz inequality:  $|\langle \phi | \psi \rangle| \leq \langle \phi | \phi \rangle^{1/2} \langle \psi | \psi \rangle^{1/2}$

Equality iff  $|\phi\rangle = \alpha |\psi\rangle$



$$|\xi\rangle = |\phi\rangle - |\delta\rangle = |\phi\rangle - \frac{|\psi\rangle \langle \psi | \phi \rangle}{\langle \psi | \psi \rangle}$$

The Schwarz inequality is simply the statement that this vector has nonnegative length, i.e.,  $\langle \xi | \xi \rangle \geq 0$ .

$$|e\rangle \equiv |\psi\rangle / \sqrt{\langle \psi | \psi \rangle} \quad |\delta\rangle = |e\rangle \langle e | \phi \rangle = \frac{|\psi\rangle \langle \psi | \phi \rangle}{\langle \psi | \psi \rangle}$$

$$\langle e | e \rangle = 1$$

$$|\phi\rangle = |\xi\rangle + |\delta\rangle, \quad \langle \xi | \delta \rangle = 0 \quad (\text{by construction})$$

Check it!

$$\langle \phi | \phi \rangle = \langle \xi | \xi \rangle + \langle \delta | \delta \rangle \geq \langle \delta | \delta \rangle = \frac{|\langle \psi | \phi \rangle|^2}{\langle \psi | \psi \rangle}$$

↑  
Notice that this is just Pythagoras's theorem

$$\Rightarrow \boxed{\langle \phi | \phi \rangle \langle \psi | \psi \rangle \geq |\langle \psi | \phi \rangle|^2}$$

Equality iff  $|\xi\rangle = 0 \iff |\phi\rangle = |\delta\rangle = |\psi\rangle \frac{\langle \psi | \phi \rangle}{\langle \psi | \psi \rangle}$   
i.e.,  $|\phi\rangle = \alpha |\psi\rangle$

Linear operators:  $A$

$$A(a|\psi\rangle + b|\phi\rangle) = aA|\psi\rangle + bA|\phi\rangle$$

Matrix elements

$$\langle e_j | A | e_k \rangle = A_{jk}$$

$A \leftrightarrow$

representation

$$\begin{pmatrix} A_{11} & A_{12} & \dots & A_{1D} \\ A_{21} & A_{22} & & \\ \vdots & & \ddots & \\ A_{D1} & & & A_{DD} \end{pmatrix}$$

$$A|\psi\rangle = \sum_j |e_j\rangle \langle e_j | A | \psi \rangle$$

$$= \sum_{j,k} |e_j\rangle \langle e_j | A | e_k \rangle \langle e_k | \psi \rangle$$

$$= \sum_{j,k} |e_j\rangle A_{jk} c_k$$

$$A|\psi\rangle \leftrightarrow \begin{pmatrix} \vdots \\ c_j \\ \vdots \\ c_D \end{pmatrix}$$

Outer-product notation:

$$A = \sum_{j,k} |e_j\rangle A_{jk} \langle e_k|$$

$$= \sum_{j,k} |e_j\rangle \langle e_j | A | e_k \rangle \langle e_k|$$

Unit operator:  $1 = I = \sum_j |e_j\rangle \langle e_j|$  (resolution of the identity)

$$AB = \sum_{j,k,l} |e_j\rangle \langle e_j | A | e_k \rangle \langle e_k | B | e_l \rangle \langle e_l|$$

$$= \sum_{j,k} |e_j\rangle \langle e_k | \sum_l A_{jl} B_{kl} \langle e_l|$$

Adjoint operator (Hermitian conjugate):

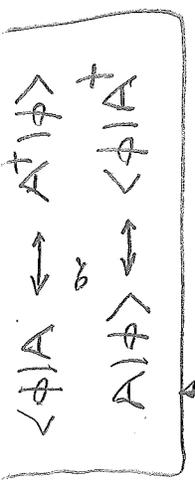
$$\langle \phi |, A | \psi \rangle = \langle A^\dagger | \phi \rangle, | \psi \rangle = \langle | \psi \rangle, A^\dagger | \phi \rangle^*$$

$$\langle \phi | A | \psi \rangle = \langle \psi | A^\dagger | \phi \rangle^*$$

no translation into Dirac notation

① Adjoint is antilinear:  $(aA + bB)^\dagger = a^* A^\dagger + b^* B^\dagger$

②  $(A^\dagger)_{jk} = \langle e_j | A^\dagger | e_k \rangle = \langle e_k | A | e_j \rangle^* = A_{kj}^* \Rightarrow (A^\dagger)^\dagger = A$



③  $(|\psi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\psi| \Rightarrow$

$A = \sum_{jk} |e_j\rangle\langle e_k| A_{jk}$

③

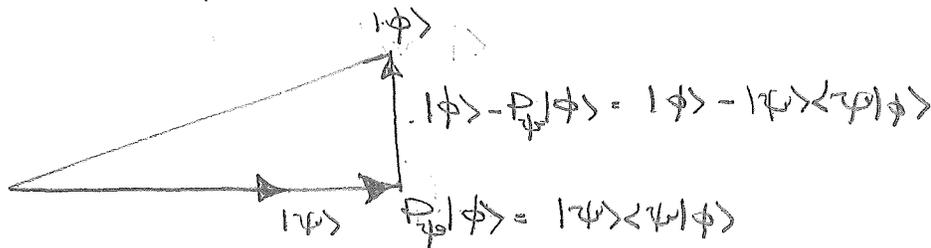
④  $(AB)^\dagger = B^\dagger A^\dagger$

$A^\dagger = \sum_{jk} |e_k\rangle\langle e_j| A_{jk}^* = \sum_{jk} |e_j\rangle\langle e_k| A_{kj}^*$

(equivalent to ③)

Projection operators:

1-d projector  $P_\psi = |\psi\rangle\langle\psi|$ ,  $\langle\psi|\psi\rangle = 1$



multi-dimensional projector: Let  $S$  be a  $d$ -dimensional subspace spanned by orthonormal vectors  $|e_j\rangle$ ,  $j=1, \dots, d$ . The projection operator onto  $S$  is

$P_S = \sum_{j=1}^d |e_j\rangle\langle e_j|$  ← unit operator in the subspace  $S$

An operator has a spectral decomposition (or eigendecomposition) if it can be written as

$A = \sum_j \lambda_j \overbrace{|e_j\rangle\langle e_j|}^{P_j} = \sum_\lambda \lambda P_\lambda$        $1 = \sum_j P_j = \sum_\lambda P_\lambda$

↑ orthonormal eigenvectors  
 $A|e_j\rangle = \lambda_j|e_j\rangle$   
 ↑ eigenvalues

All vectors in  $S_\lambda$  are eigenvectors with eigenvalue  $\lambda$ ; they are said to be degenerate.  
 Eigenvectors with different eigenvalues are orthogonal; they are nondegenerate.

Support  $S$  is the subspace spanned by eigenvectors with nonzero eigenvalue. Rank is the dimension of  $S$ .  
Null subspace (kernel)  $N$  is subspace spanned by eigenvectors with zero eigenvalue.  $S$  and  $N$  are orthogonal.

$P_\lambda = \sum_{j=\lambda} |e_j\rangle\langle e_j|$  is the projector onto the subspace  $S_\lambda$  spanned by eigenvectors with eigenvalue  $\lambda$ ;  
eigenprojector onto eigensubspace  $S_\lambda$

Commutator:  $[A, B] = AB - BA$

⊕

Normal operator  $M$ :  $[M, M^\dagger] = 0$

Theorem. An operator has a spectral decomposition

iff it is normal.

Proof: s.d.  $\Rightarrow$  normal (this is trivial)

s.d.  $\Leftarrow$  normal (N&C Box 2.2)

Hermitian operators:  $H = H^\dagger$

The eigenvalues and eigenvectors of non-normal operators are complicated. They make up a major topic in linear algebra, which we don't have to worry about.

Hermitian operators are normal operators whose eigenvalues are real.

Only need  $U^\dagger U = 1$  in finite dimensions.  
 $U^\dagger U = 1 \Rightarrow (UU^\dagger)^\dagger = UU^\dagger$   
 $\Rightarrow UU^\dagger = P$   
 $\text{tr}(P) = \text{tr}(UU^\dagger) = \text{tr}(U^\dagger U) = \text{tr}(I) = D$   
 $\Rightarrow I = P = UU^\dagger$

Unitary operators:  $U^\dagger U = 1 = UU^\dagger$

Unitary operators are normal operators whose eigenvalues are phases.

①  $U$  preserves inner products:

$$\langle \psi | U^\dagger (U | \phi \rangle) = \langle \psi | U^\dagger U | \phi \rangle = \langle \psi | \phi \rangle$$

$\Rightarrow$  ②  $U$  takes orthonormal bases to orthonormal bases:

Basis change  $\rightarrow$

$$|f_j\rangle = U |e_j\rangle \iff U = \sum_j |f_j\rangle \langle e_j|$$

$$= \sum_k |e_k\rangle \langle e_k | U |e_j\rangle$$

$$= \sum_k |e_k\rangle U_{kj} \rightarrow$$

(unitary) transformation matrix between the bases.

$$= \sum_k |e_k\rangle \langle e_j| U_{kj}$$

rows and columns of  $U_{jk}$  are orthonormal vectors

$$UU^\dagger = 1 \iff \delta_{jk} = \sum_l U_{jl} (U^\dagger)_{lk} = \sum_l U_{jl} U_{kl}^*$$

Projection operators are Hermitian operators all of whose eigenvalues are 0 or 1. A Hermitian operator  $P$  is a projection operator iff  $P^2 = P$ .

Operator functions: A function  $f: \mathbb{C} \rightarrow \mathbb{C}$  can be extended to a function on normal operators by

$$f(A) = f\left(\sum_j \lambda_j |e_j\rangle\langle e_j|\right) = \sum_j f(\lambda_j) |e_j\rangle\langle e_j|$$

Example:  $U = \sum_j e^{i\phi_j} |e_j\rangle\langle e_j|$ . Define a Hermitian operator  $H = \sum_j \phi_j |e_j\rangle\langle e_j|$ . Then

$$U = \sum_j e^{i\phi_j} |e_j\rangle\langle e_j| = e^{iH} \quad \leftarrow \text{Any unitary can be written like this for some } H=H^\dagger$$

Trace: For any operator  $A$ ,

$$\left(\begin{array}{l} \text{trace} \\ \text{of } A \end{array}\right) = \text{tr}(A) \equiv \sum_j \langle e_j | A | e_j \rangle$$

①  $\text{tr}$  is linear:  $\text{tr}(aA + bB) = a\text{tr}(A) + b\text{tr}(B)$

② Basis-independence:

$$\begin{aligned} \sum_j \langle e_j | A | e_j \rangle &= \sum_{j,k} \langle e_j | f_k \rangle \langle f_k | A | e_j \rangle \\ &\quad \uparrow \text{insert resolution of } 1 \\ &= \sum_{j,k} \langle f_k | A | e_j \rangle \langle e_j | f_k \rangle \\ &= \sum_k \langle f_k | A | f_k \rangle \\ &\quad \uparrow \text{remove resolution of } 1 \end{aligned}$$

$$\begin{aligned}
 \textcircled{3} \quad \text{tr}(AB) &= \sum_{j,k} \langle e_j | A | e_k \rangle \langle e_k | B | e_j \rangle \\
 &= \sum_{j,k} \langle e_k | B | e_j \rangle \langle e_j | A | e_k \rangle \\
 &= \text{tr}(BA)
 \end{aligned}$$

Insert and  
remove resolution  
of  $\mathbb{1}$

$$\textcircled{4} \quad \text{Cyclic property of trace: } \text{tr}(ABC) = \text{tr}(CAB)$$

$$\begin{aligned}
 \textcircled{5} \quad \text{tr}(|\psi\rangle\langle\phi|) &= \sum_j \langle e_j | \psi \rangle \langle \phi | e_j \rangle \\
 &= \sum_j \langle \phi | e_j \rangle \langle e_j | \psi \rangle
 \end{aligned}$$

$$= \langle \phi | \psi \rangle$$

remove resolution  
of  $\mathbb{1}$

$$\textcircled{6} \quad \text{tr}(A|\psi\rangle\langle\phi|) = \text{tr}(|\psi\rangle\langle\phi|A) = \langle\phi|A|\psi\rangle$$

$\text{tr}$  turns outer product into inner product

Simultaneous eigenvectors Theorem. Two normal operators have simultaneous eigenvectors iff they commute.

Proof:

⇒ Trivial

⇐ Let A and B be normal operators such that  $[A, B] = 0$ . A has eigendecomposition

$$A = \sum_{\lambda} \lambda P_{\lambda}$$

↙ projector onto eigensubspace  $S_{\lambda}$

Any vector  $|e\rangle$  in  $S_{\lambda}$  is an eigenvector of A with eigenvalue  $\lambda$ , i.e.,  $A|e\rangle = \lambda|e\rangle$ .

$$A(B|e\rangle) = B(A|e\rangle) = \lambda B|e\rangle$$

↑  $[A, B] = 0$

So  $B|e\rangle$  is also an eigenvector of A with eigenvalue  $\lambda$  and thus is in  $S_{\lambda}$ . This means B maps each eigensubspace of A into itself, so we can write it as  $B = \sum_{\lambda} B_{\lambda}$ , where  $B_{\lambda}$  operates in  $S_{\lambda}$ . Diagonalizing the  $B_{\lambda}$ 's gives eigenvectors of B that lie in the subspaces  $S_{\lambda}$  and thus are also eigenvectors of A.

Notice that if B-eigenvectors from different  $S_{\lambda}$  are degenerate, we can superpose them to make B-eigenvectors that do not lie in any  $S_{\lambda}$ .

# "Axioms" of quantum mechanics.

Pure states

1. Quantum states. The state of a quantum system is a ray in Hilbert space. A ray is the collection of states  $a|\psi\rangle$ ,  $a \in \mathbb{C}$  ( $a|\psi\rangle$  is the same state as  $|\psi\rangle$ ). The set of rays is called projective Hilbert space.

ⓐ We generally normalize states so that  $\langle\psi|\psi\rangle = 1$ , but it is still true that  $e^{i\phi}|\psi\rangle$  is the same state as  $|\psi\rangle$ .  
 ↑ global phase vs. relative phase in a superposition

A normalized vector is called a state vector.

ⓑ We can get rid of the global phase freedom by identifying states with 1-d projectors  $|\psi\rangle\langle\psi|$ .  
pure states

Von Neumann measurement statistics

2. Observables. An observable is a Hermitian operator  $A = \sum_j \lambda_j |e_j\rangle\langle e_j| = \sum_j \lambda_j P_j$ . The result of a measurement of  $A$  is one of the eigenvalues  $\lambda$ . When the system is in state  $|\psi\rangle$ , the probability of getting eigenvalue  $\lambda$  is

$$P(\lambda) = \langle\psi|P_\lambda|\psi\rangle = \sum_{j:\lambda_j=\lambda} |\langle e_j|\psi\rangle|^2$$

$$= \text{tr}(P_\lambda |\psi\rangle\langle\psi|)$$

probability for eigenvector  $|e_j\rangle$

$\langle e_j|\psi\rangle = c_j$   
 is a probability amplitude

Eigenvalues label the measurement results. Probabilities only depend on the eigenvectors. We often speak of a measurement "in the basis  $|e_j\rangle$ " and say the result is  $|e_j\rangle$ .

von Neumann measurement

3. Post-measurement states. A measurement with result  $|e_j\rangle$  leaves the system in the state  $|e_j\rangle$ :

$$|\psi\rangle \rightarrow \frac{P_j |\psi\rangle}{\langle \psi | P_j | \psi \rangle^{1/2}} = \frac{P_j |\psi\rangle}{p_j}$$

"collapse of the wave function"

Dynamics of an isolated system

4. Dynamics. The system has a Hamiltonian  $H$ . If the system is isolated, its state vector changes according to the Schrödinger equation:

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle \implies |\psi(t)\rangle = \underbrace{e^{-iHt/\hbar}}_{\text{unitary evolution operator } U(t)} |\psi(0)\rangle$$

[Many people working in quantum info theory forget that unitary evolution comes from system Hamiltonians.]

Expectation values:  $\langle A \rangle = \sum_{\lambda} \lambda p(\lambda)$

$$= \langle \psi | \underbrace{\sum_{\lambda} \lambda P_{\lambda}}_A | \psi \rangle$$

$$= \langle \psi | A | \psi \rangle$$

All these will have to be generalized as we proceed.