To: *Information Physics Group*
From: *C. M. Caves*
Subject: **Heisenberg and Schrödinger pictures in linear optics**
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One usually describes a linear-optics device as transforming a set of N input modes, with modal annihilation operators a_j, being transformed into a set of N output modes, with modal annihilation operators b_j. This is a Heisenberg-picture description.

In the Heisenberg picture the state of the field doesn't change, whereas the operators transform according to $A \to U^\dagger A U \equiv B$, A being the input operator and B being the output operator. Applied to the modal annihilation operators, this transformation becomes

$$a_j \to U^\dagger a_j U = b_j \ . \tag{1}$$

The assumption of linear optics is that the output operators are linearly related to the input operators:

$$b_j = \sum_k M_{jk} a_k \qquad \Longleftrightarrow \qquad b_j^\dagger = \sum_k M_{jk} a_k^\dagger \ . \tag{2}$$

It is often convenient to gather the annihilation and creation operators into column vectors,

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_N \end{pmatrix} \qquad \text{and} \qquad \mathbf{a}^\dagger = \begin{pmatrix} a_1^\dagger \\ \vdots \\ a_N^\dagger \end{pmatrix} , \tag{3}$$

and to write the transformation equations as matrix equations,

$$\mathbf{b} = U^\dagger \mathbf{a} U = M\mathbf{a} \qquad \Longleftrightarrow \qquad \mathbf{b}^\dagger = U^\dagger \mathbf{a}^\dagger U = M^* \mathbf{a}^\dagger \ . \tag{4}$$

The preservation of commutators constrains the transformation matrix M to be unitary:

$$\delta_{jk} = [b_j, b_k^\dagger] = \sum_{l,m} M_{jl} M_{km}^* \underbrace{[a_l, a_m^\dagger]}_{=\,\delta_{lm}} = \sum_l M_{jl} M_{kl}^* \qquad \Longleftrightarrow \qquad M^\dagger M = 1 \ . \tag{5}$$

This allows us to invert the transformations (4):

$$\mathbf{a} = M^\dagger \mathbf{b} \qquad \Longleftrightarrow \qquad \mathbf{a}^\dagger = M^T \mathbf{b}^\dagger \ . \tag{6}$$

The initial field state can always be written as

$$|\psi\rangle = f(\mathbf{a}^\dagger)|0\rangle \ , \tag{7}$$

where $f(\mathbf{a}^\dagger)$ is some function of the modal creation operators. Since the state doesn't change in the Heisenberg picture, we get the description of the output by rewriting the same state in terms of output operators:

$$|\psi\rangle = f(M^T \mathbf{b}^\dagger)|0\rangle \ . \tag{8}$$

How does all this look from the Schrödinger picture? All we need to do is to rewrite the transformations (4) as

$$U\mathbf{a}U^\dagger = M^\dagger\mathbf{a} \qquad \Longleftrightarrow \qquad U\mathbf{a}^\dagger U^\dagger = M^T\mathbf{a}^\dagger \; . \tag{9}$$

In the Schrödinger picture the input state changes to an output state

$$U|\psi\rangle = Uf(\mathbf{a}^\dagger)U^\dagger \underbrace{U|0\rangle}_{=\,|0\rangle} = f(U\mathbf{a}^\dagger U^\dagger)|0\rangle = f(M^T\mathbf{a}^\dagger)|0\rangle \; , \tag{10}$$

where we use the fact that a linear-optics transformation leaves the vacuum unchanged (up to a phase, which we can choose to be zero). Comparison of Eqs. (8) and (10) shows that the two pictures give the same description of the output.