Lecture # 11: Quantum Trajectories (I)

In our studies we have seen the need for the density operator to specify a quantum state in two different circumstances:

(i) State prepared "statistically" according to a classical probability distribution of different pure states:

$$\hat{\rho} = \sum_i P_i \, |\psi_i\rangle\langle\psi_i|$$

(ii) State is a "subsystem" of a larger set of degrees of freedom, and they are <u>entangled</u>

$$\hat{\rho}_S = \mathrm{Tr}_E \left(|\Psi_{SE}\rangle\langle\Psi_{SE}| \right)$$

Are these two pictures related? In a certain sense, yes. We can think of the environment as performing a measurement of the system but not telling us the result. This is equivalent to "Alice" preparing a state for "Bob", but not telling him the result, only the probabilities he should expect. The interpretation, of course is steeped in the deep issues of quantum measurement theory, but this need not concern us now.

Measurements of a quantum system cause "back-action" or "collapse" of the state, with associated randomness.

A continuously measured quantum state will thus evolve stochastically. This evolution has come to be known as a "quantum trajectory", a term originally coined by Howard Carmichael. The density operator is then the averaged over many different trajectories.

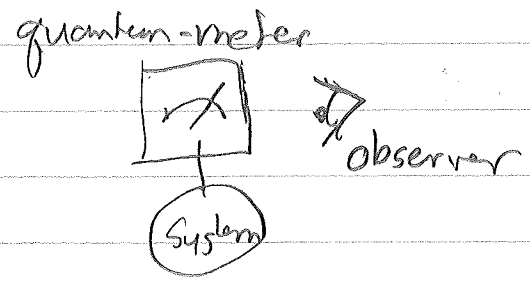
The development of stochastic wave function evolution proceeded in parallel along a number of fronts in the late 1980's to early 1990's and falls under many names including:

- Quantum Monte Carlo Wave functions
- Quantum Jump method
- Quantum State Diffussion
- Quantum Trajectories

Today we understand the connections between these approaches based on modern quantum measurement theory. We begin with a brief overview of the general theory of quantum measurement

Introduction to Quantum Measurement

A fully quantum theory of measurement was proposed by von Neumann. A quantum-meter is coupled ~~to~~ the quantum-system. The meter's degrees of freedom the get displaced according to the value of the observable we want to measure.



Von Neumann paradigm

<u>Model:</u> $\hat{A}_s$ is a Hermitian operator for on S

Interaction Hamiltonian: $\hat{H}_{SM} = \hbar K \hat{A}_s \hat{P}_M$

where $\hat{P}_M$ is the generator of displacements of the meter position $\hat{X}_M$.

Suppose the system is in a quantum state $|\psi_s\rangle$ and the meter is at the origin $|0\rangle_M$

then after the interaction:

$$|\psi_s\rangle|0_M\rangle \Rightarrow e^{-iK\hat{A}_s\hat{P}_M} \sum' c_a \underbrace{|a_s\rangle}_{\text{eigenstates of } \hat{A}_s} \otimes |0_M\rangle$$

$$= \sum_a' c_a |a_s\rangle \otimes |X_M = Ka_s\rangle$$

Now the meter is observed. We can measure the position of the meter with fine resolution, then where we find $\underline{X_M = Ka_s}$ the system is in $\underline{|a_s\rangle}$. This occurs with probability $P_{a_s} = |c_a|^2$.

We thus see a model of a <u>projective measurement</u>

$$|\psi_s\rangle \Rightarrow |a_s\rangle \text{ with probability } |c_a|^2$$

Under a projective measurement, we have two rules

- Probability of measurement outcome:

$$p_a = \langle \psi_s | \hat{\mathbb{P}}_a | \psi_s \rangle = |\langle a | \psi_s \rangle|^2$$

- Post-measurement state conditioned on measurement result:

$$|\psi_s\rangle \Rightarrow \frac{\hat{\mathbb{P}}_a |\psi_s\rangle}{\| \hat{\mathbb{P}}_a |\psi_s\rangle \|} = \frac{\hat{\mathbb{P}}_a |\psi_s\rangle}{\sqrt{p_a}} = |a\rangle$$

where $\hat{\mathbb{P}}_a = |a\rangle\langle a|$ are projectors on eigen-space

The projectors form a resolution of the identity

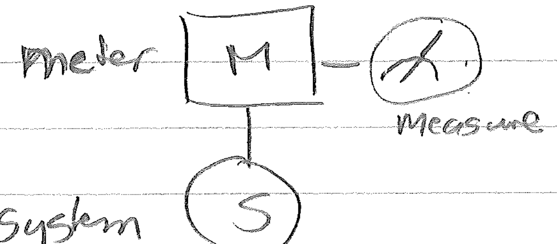
$$\sum_a \hat{\mathbb{P}}_a = \hat{\mathbb{1}}_s$$

$$\Rightarrow \sum_a p_a = 1 \quad \text{(normalization of probabilities)}$$

The von Neumann paradigm is an example of an indirect quantum measurement. The meter is a quantum subsystem unto itself. It become quantum-correlated with the system to be measured (entangled). When the observer measures the meter, he/she learns about the system and thus updates his/her state assignment about the system conditioned on the result. Note: This doesn't solve the "measurement problem". There is no "collapse" until the observer looks at the meter. How only one outcome is seen is not resolved within quantum theory.

Generalized Measurement

Underlying the von Neumann paradigm was the assumption that the meter state $|X_m\rangle$ and $|X'_m\rangle$ are perfectly distinguishable, i.e. $\langle X'_m | X_m\rangle = \delta(X_m - X'_m)$ (meter state are orthogonal for $a \neq a'$). We can generalize beyond projective measure when the meter states are not orthogonal. We have the formalism for doing this based on our discussion of the C-P Map.



Bipartite coupling of system and meter followed by measurement of the meter.

If we don't "observe" the meter, we know that the system state evolves according to a completely positive map, described by Kraus decomposition. Let us take the case that S is initially in a pure state.

$$\hat{\rho}^{in} = |\psi_S\rangle\langle\psi_S| \Rightarrow \hat{\rho}^{out} = \sum_{\mu} \hat{M}_{\mu} |\psi_S\rangle\langle\psi_S| \hat{M}^{\dagger}_{\mu}$$

where $\hat{M}_{\mu} = \langle\mu_M| \hat{U}_{SM} |0_M\rangle$ is a Kraus operator

given by the partial matrix element of the System-Meter entangling unitary between the initial "fiducial" state of the meter and a basis state $|\mu_M\rangle$ for some complete (maybe over complete) basis of M.

Now we have a way to interpret the C.P map.

Let $$|\psi_\mu\rangle = \frac{\hat{M}_\mu|\psi_s\rangle}{\|\hat{M}_\mu|\psi_s\rangle\|} \equiv \frac{\hat{M}_\mu|\psi_s\rangle}{\sqrt{p_\mu}}$$

where $$p_\mu = \|\hat{M}_\mu|\psi_s\rangle\|^2 = \langle\psi_s|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi_s\rangle$$

$$\Rightarrow \hat{\rho}_{out} = \sum_\mu p_\mu |\psi_\mu\rangle\langle\psi_\mu|$$

The output state is generally mixed because we have thrown out the measurement record stored in the meter M, and thus $\hat{\rho}_{out}$ is a statistical mixture weighted by the probability of different outcomes μ.

If on the other hand, we observe the meter and find it in state $|\mu_M\rangle$, then conditioned on this result, we infer the system to be in the state

$$|\psi_\mu\rangle = \frac{\hat{M}_\mu|\psi_s\rangle}{\|\hat{M}_\mu|\psi_s\rangle\|}$$

This occurs with probability $p_\mu = \langle\psi_s|\hat{M}_\mu^\dagger \hat{M}_\mu|\psi_s\rangle$

IMPORTANT NOTES:

- If $\hat{U}_{SM}$ is entangling, then $\hat{M}_\mu$ is generally not the identity $\Rightarrow$ Quantum backaction on $|\psi_s\rangle$ conditioned of μ
- The Kraus operators are generally not projectors, so this is generally not a projective measurement
- The indirect measurement gives a generalization the dictates the probability of outcomes and the post-measurement state.

Important Notes Continued

- In a measurement model $\{|\mu_\mu\rangle\}$ is assumed to be a complete basis, so the map is trace preserving

$$\Rightarrow \sum_\mu \hat{M}_\mu^\dagger \hat{M}_\mu = \hat{\mathbb{1}}_S$$

We define
$$\hat{E}_\mu \equiv \hat{M}_\mu^\dagger \hat{M}_\mu$$
$$\Rightarrow \sum_\mu \hat{E}_\mu = \hat{\mathbb{1}}_S$$

The set of operators $\{\hat{E}_\mu\}$ are said to form a "POVM" (positive operator-valued measure). A POVM is a set of positive operators that form a resolution of the identity. Projectors form a POVM, but $\{\hat{E}_\mu\}$ are generally not projectors

For a generalized measurement described by a POVM, $p_\mu = \langle\psi_S|\hat{E}_\mu|\psi_S\rangle$

Note: A POVM can be considered the fundamental object in quantum measurement theory. The post-measurement state, however is only determined given a measurement model with Kraus operators.

Generalization to input mixed states

$$p_\mu = \mathrm{Tr}(\hat{E}_\mu \hat{\rho}_{in}) \qquad \hat{\rho}_{out}^{\mu} = \frac{\hat{M}_\mu \hat{\rho}_{in} \hat{M}_\mu^\dagger}{p_\mu}$$

The Master Equation and Continuous Measurement

Consider now a differential map on the density operator described by a Lindblad equation.

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_{\mu}\left(-\frac{1}{2}(\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}\hat{\rho} + \hat{\rho}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}) + \hat{L}_{\mu}\hat{\rho}\hat{L}_{\mu}^{\dagger}\right)$$

We have seen that this evolution can be viewed as a differential map:

$$\hat{\rho}(t+dt) = \hat{M}_0(dt)\,\hat{\rho}(t)\,\hat{M}_0(dt) + \sum_{\mu>0}\hat{M}_{\mu}(dt)\,\hat{\rho}(t)\,\hat{M}_{\mu}^{\dagger}(dt)$$

where $\hat{M}_{\mu}(dt) = \sqrt{dt}\,\hat{L}_{\mu}$

$$\hat{M}_0(dt) = \hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}_{eff}\,dt$$

Here I have written $\hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\sum_{\mu}\hat{L}_{\mu}^{\dagger}\hat{L}_{\mu}$ as the "effective Hamiltonian".
It it a non-Hermitian operator whose "real part" is the Hamiltonian and whose "imaginary part" accounts for decay as we will see.

Suppose that at time t the state is pure, $|\psi(t)\rangle$.
Then according to this map, at time $t+dt$

$$\rho(t+dt) = \hat{M}_0(dt)|\psi(t)\rangle\langle\psi(t)|\hat{M}_0^{\dagger}(dt) + \sum_{\mu>0}\hat{M}_{\mu}(dt)|\psi(t)\rangle\langle\psi(t)|\hat{M}_{\mu}^{\dagger}(dt)$$

$$= (\hat{\mathbb{1}} - \frac{i}{\hbar}\hat{H}_{eff}dt)|\psi(t)\rangle\langle\psi(t)|(\hat{\mathbb{1}} + \frac{i}{\hbar}\hat{H}_{eff}^{\dagger}dt)$$

$$+ dt\sum_{\mu}\hat{L}_{\mu}|\psi(t)\rangle\langle\psi(t)|\hat{L}_{\mu}^{\dagger}$$

Based on the discussion of a C.P map as the environment performing a measurement of the system and then throwing away the record, we see the following interpretation of the differential map.

– With probability $p_\mu = \langle\psi(t)|\hat{M}_\mu^\dagger(dt)\hat{M}_\mu(dt)|\psi(t)\rangle$

$$= dt\,\langle\psi(t)|\hat{L}_\mu^\dagger\hat{L}_\mu|\psi(t)\rangle$$

The state "jumps" from $|\psi(t)\rangle \Rightarrow \frac{\hat{M}_\mu|\psi\rangle}{\|\hat{M}_\mu|\psi\rangle\|} = \frac{\hat{L}_\mu|\psi(t)\rangle}{\|\hat{L}_\mu|\psi(t)\rangle\|}$

– With probability $p_0 = \langle\psi(t)|\hat{M}_0^\dagger(dt)\hat{M}_0(dt)|\psi(t)\rangle$

$$p_0 = \langle\psi(t)|(\hat{\mathbb{1}} + \tfrac{i}{\hbar}\hat{H}_{eff}^\dagger dt)(\hat{\mathbb{1}} - \tfrac{i}{\hbar}\hat{H}_{eff}\,dt)|\psi(t)\rangle$$

$$= \langle\psi(t)|(\mathbb{1} + \tfrac{i}{\hbar}(\hat{H}_{eff} + \hat{H}_{eff}^\dagger)dt|\psi(t)\rangle = 1 - dt\langle\psi(t)|\sum_\mu L_\mu^\dagger L_\mu|\psi(t)\rangle$$

$$= 1 - \sum_\mu p_\mu$$

There is no-jump. Under the no jump case the state evolves according to

$$|\psi(t)\rangle \Rightarrow \frac{\hat{M}_0(dt)|\psi(t)\rangle}{\|\hat{M}_0(dt)|\psi(t)\rangle\|} = \frac{e^{-i\hat{H}_{eff}dt/\hbar}|\psi(t)\rangle}{\|e^{-i\hat{H}_{eff}dt/\hbar}|\psi(t)\rangle\|}$$

Where I have used $e^{-i\hat{H}_{eff}dt/\hbar} \approx \mathbb{1} - \frac{i}{\hbar}\hat{H}_{eff}\,dt$ to $O(dt)$

This provides an algorithm for calculating the time-evolution of $\hat{\rho}(t)$ through a stochastic (Monte-Carlo) algorithm. Suppose $\hat{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|$

- The density operator is a statistical mixture of pure states $|\psi(t)\rangle$
- Each pure state evolves "stochastically"
- In a time interval dt

$$|\psi(t+dt)\rangle = \begin{cases} \dfrac{\hat{L}_\mu|\psi(t)\rangle}{\| \hat{L}_\mu|\psi(t)\rangle\|} & \text{with prob } p_\mu = dt\,\langle\psi(t)|\hat{L}_\mu^\dagger\hat{L}_\mu|\psi(t)\rangle \\[2ex] \dfrac{e^{-\frac{i\hat{H}_{eff}dt}{\hbar}}|\psi(t)\rangle}{\| e^{-\frac{i\hat{H}_{eff}dt}{\hbar}}|\psi(t)\rangle\|} & \text{with prob } p_0 = 1-\sum_\mu p_\mu \end{cases}$$

- We simulate the probability distribution by sampling with the statistics above over many relations of the stochastic evolution of $|\psi(t)\rangle$

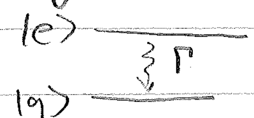
The $\hat{\rho}(t) = \lim_{N\to\infty} \frac{1}{N}\sum_{i^{th}\text{ realization}} |\psi_i(t)\rangle\langle\psi_i(t)|$

This procedure as detailed here is call the "quantum-Monte-Carlo-Wave Function" simulation of the Master equation, original discussed in K. Mølmer, Y. Castin, and J. Dalibard PRL

Numerically, this is useful if the dimension of the Hilbert is very large and memory is expensive, as $\hat{\rho}$ required d^2 element by $|\psi(t)\rangle$ only d.
The procedure allow has fundamental implications

Physical Example: Decay of Two-Level Atom

Consider the simplest problem of a two-level atom decaying in the vacuum (zero-temp reservoir).



The Master Equation with one Lindblad op $\hat{L} = \sqrt{\Gamma dt}\,\hat{\sigma}$

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\Gamma}{2}(\hat{\sigma}_+ \hat{\sigma}_- \hat{\rho} + \hat{\rho}\,\hat{\sigma}_+ \hat{\sigma}_-) + \Gamma \hat{\sigma}_- \hat{\rho}\,\hat{\sigma}_+$$

where $\hat{H} = E_e |e\rangle\langle e| = E_e\,\hat{\sigma}_+ \hat{\sigma}_-$

This can be rewritten as

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]' + \Gamma \hat{\sigma}_- \hat{\rho}\,\hat{\sigma}_+$$

where $\hat{H}_{eff} = \left(E_e - i\frac{\hbar\Gamma}{2}\right)|e\rangle\langle e|$

$$[\hat{H}_{eff}, \hat{\rho}]' \equiv \hat{H}_{eff}\hat{\rho} - \hat{\rho}\,\hat{H}_{eff}^{\dagger}$$

If at time t $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$\hat{\rho}(t+dt) = \left(1 - \frac{i}{\hbar}\hat{H}_{eff}\,dt\right)|\psi(t)\rangle\langle\psi(t)|\left(1 + \frac{i}{\hbar}\hat{H}_{eff}^{\dagger}\,dt\right) + \sqrt{\Gamma dt}\,\hat{\sigma}_-|\psi(t)\rangle\langle\psi(t)|\sqrt{\Gamma dt}\,\hat{\sigma}_+$$

$$= (1 - dp(t))\,|\phi_0\rangle\langle\phi_0| + dp(t)\,|\phi_1\rangle\langle\phi_1|$$

where $dp(t) = \langle\psi(t)|\hat{L}^{\dagger}\hat{L}|\psi(t)\rangle = \Gamma dt\,\langle\psi(t)|\hat{\sigma}_+\hat{\sigma}_-|\psi(t)\rangle$

$$= \Gamma\,dt\,\langle\psi(t)|e\rangle\langle e|\psi(t)\rangle = \Gamma|\langle e|\psi(t)\rangle|^2 dt$$

$$= P_e(t)\,\Gamma\,dt$$

$$|\phi_1(t)\rangle = \frac{\hat{L}\,|\psi(t)\rangle}{\|\hat{L}\,|\psi(t)\rangle\|} = \frac{\hat{\sigma}_-\,|\psi(t)\rangle}{\|\hat{\sigma}_-\,|\psi(t)\rangle\|}$$

$$|\phi_0\rangle = \frac{e^{(-i\omega_{eg} - \frac{\Gamma}{2})dt\,|e\rangle\langle e|}\,|\psi(t)\rangle}{\|e^{(-i\omega_{eg} - \frac{\Gamma}{2})dt\,|e\rangle\langle e|}\,|\psi(t)\rangle\|}$$

We thus see that in time $t \to t+dt$

- With probability $dp(t) = P_e(t)\Gamma dt$ the system undergoes a "quantum jump" correlated with the spontaneous photon into the environment

$$|\psi(t)\rangle \Rightarrow \frac{\hat{\sigma}_-\,|\psi(t)\rangle}{\|\hat{\sigma}_-\,|\psi(t)\rangle\|} = |g\rangle\langle e|\frac{\left(c_g(t)|g\rangle + c_e(t)|e\rangle\right)}{|c_e(t)|}$$

$$= |g\rangle \quad \text{(up to overall phase)}$$

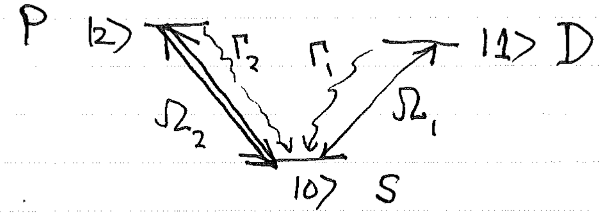
- With probability $1 - dp(t) = \|e^{-iH_{eff}dt/\hbar}\,|\psi\rangle\|$ the atom does not decay, and there is no spontaneous emission. Under this case the state evolves continuously according to the non-Hermitian effective Hamiltonian

$$|\psi(t+dt)\rangle = \frac{e^{-i\hat{H}_{eff}\frac{dt}{\hbar}}\,|\psi(t)\rangle}{\|e^{-iH_{eff}dt/\hbar}\,|\psi(t)\rangle\|}$$

The evolution of $|\psi(t)\rangle$ under the no-jump case is essential and has an important physical interpretations.

The notion of quantum jumps has played an important role in the developement of the notion quantum trajectories.

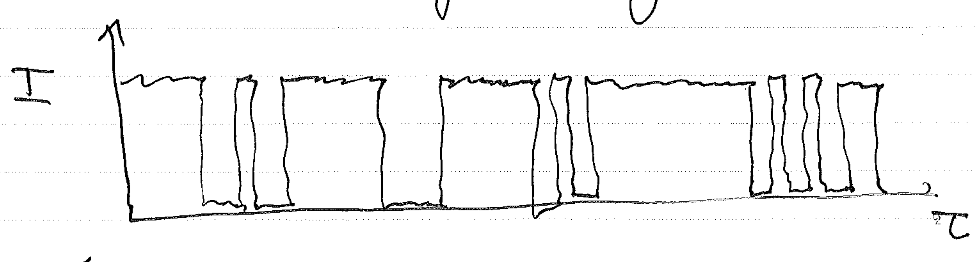
Consider a "V-system"



- $|0\rangle - |2\rangle$ is a strong dipole allowed transition with rapid decay rate Γ_2
- $|0\rangle - |1\rangle$ is a weak quadrapole transition with very slow (meta-stable) decay Γ_1

In 1975 Dehmelt proposed such a system to measure the very narrow line width Γ_1. Termed "electron shelving", the strong transition "amplifies" the weak transition. Every time state $|1\rangle$ decays (say every second), 10^9 photons/sec are scattered on the $|0\rangle \to |2\rangle$ transition.

In 1985 Kimble and Cook showed how the fluorescence signal at ω_{02} ~~would exhibit~~ for a single atom would exhibit a "random telegraph" signal with the atom randomly blinking on and off



This was observed by Winland's group ~~has~~ in 1986 for a single traped Hg atom

Every time the signal is turned on, it is triggered by a "quantum jump" from $|0\rangle \to |1\rangle$. Zoller's analysis in 1987 was one path to quantum trajectories.

Evolution under the no jump condition

Under the condition that no jump occurs, the wave function evolves under the action of a non-Hermitian Hamiltonian. How do we interpret this?

Consider the simplest case of the atom coupled to the vacuum in the absense of a driving laser field.

Suppose at time t $|\psi(t)\rangle = c_g|g\rangle + c_e|e\rangle$

$$\Rightarrow |\tilde{\psi}(t+\delta t)\rangle = e^{-\frac{i}{\hbar}H_{eff}\delta t}|\tilde{\psi}(t+\delta t)\rangle$$

$$= c_g|g\rangle + e^{-i\omega_{eg}\delta t - \frac{\Gamma}{2}\delta t} c_e|e\rangle$$

$$\Rightarrow |\psi(t+\delta t)\rangle = \frac{|\tilde{\psi}(t+\delta t)\rangle}{\|\tilde{\psi}(t+\delta t)\|} = c_g\left(1 + \frac{\Gamma}{2}\delta t|c_e|^2\right)|g\rangle$$

$$+ e^{-i\omega_{eg}\delta t}\left(1 - \frac{\Gamma\delta t}{2}|c_g|^2\right)c_e|e\rangle$$

(to first order in $\Gamma\delta t$)

Thus we see that in addition to the phase evolution there is a small rotation of the state from $|e\rangle$ to $|g\rangle$. This has the following interpretation.

No jump corresponds to no detection of a photon in the vacuum. this null detection gives us some information. With this information we must readjust our quantum state (which represents our knowledge of the atom). Not seeing a photon, we are more likely to have a ground state atom than an excited state atom, hence the decay of the excited state probability and growth of the ground states amplitude

The rotation of the state associated with a null detection is essential to get proper statistics.

If it did not occur than the probability for observing a jump would be constant with time

$$dp(t) = P_e(t)\,\Gamma\,dt = |c_e|^2\,\Gamma\,dt = \text{constant}$$

Thus if $|\psi(0)\rangle = c_g|g\rangle + c_e|e\rangle$ we would always see a jump at some point, whereas we know from ~~quan~~ quantum mechanics that given this initial state we expect no jump from $t=0 \to \infty$ with probability $|c_g|^2$. The rotation for null measurement ensures this.

If no jump occurs from $0 \to t$ we have

$$|\psi(t)\rangle = \frac{c_g|g\rangle + e^{-i\omega_e t}\,c_e e^{-\Gamma t/2}\,|e\rangle}{\sqrt{|c_g|^2 + |c_e|^2 e^{-\Gamma t}}}$$

$\Rightarrow$ Probability for no jump between 0 and t $P(t)$ satisfies

$$P(t+\delta t) = \left(P_{\text{no jump } t \to t+\delta t}\right)\cdot P_{\text{no jump } 0 \to t}$$

$$= \left(1 - |\langle e|\psi(t)\rangle|^2\,\Gamma\,\delta t\right) P(t)$$

$$= \left(1 - \Gamma\,\delta t\,\frac{|c_e|^2 e^{-\Gamma t}}{(|c_g|^2 + |c_e|^2 e^{-\Gamma t})}\right) P(t)$$

$$\Longrightarrow \boxed{P(t) = |c_g|^2 + e^{-\Gamma t}\,|c_e|^2}$$

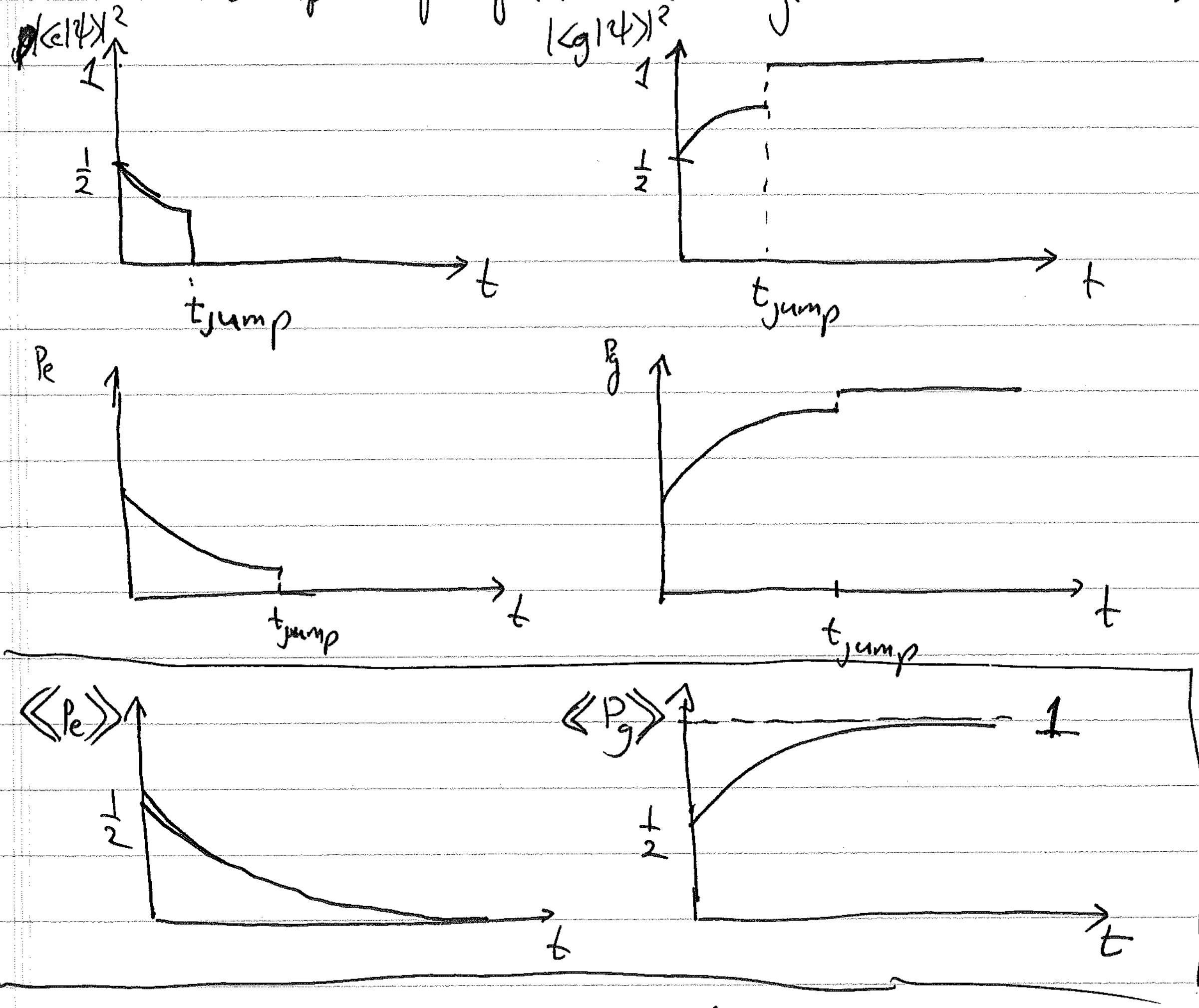
As $t \to \infty$ $\quad P \to |c_g|^2$ as expected

Example: $|\psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle + |e\rangle\right)$

We know the simple solution to the master eq

$$\begin{cases} P_e(t) = \rho_{ee}(t) = \frac{1}{2} e^{-\Gamma t} \\ P_g(t) = \rho_{gg}(t) = 1 - \frac{1}{2} e^{-\Gamma t} \\ \rho_{eg}(t) = \frac{1}{2} e^{-i\omega_{eg} t - \frac{\Gamma}{2} t} \end{cases}$$

A sample of quantum trajectories is sketched



Ensemble averaged behavior over $N \to \infty$ trajectories with t_{jump} arising from Monte-Carlo wavefunction algorithm reproduces the density matrix behaviour.