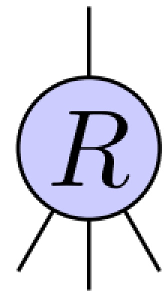
Tensor networks in quantum information

Rafael Alexander
CQuIC Postdoctoral Fellow

What is it?

- A diagrammatic method for representing tensors

$$R^{\rho}{}_{\sigma\mu\nu} \Longrightarrow$$



- Why is this useful?
- Gives a convenient way of bookkeeping how much entanglement there is between different parts of a quantum system
- Has resulted in many useful numerical and analytic methods for analysing interesting many-body quantum states.

Goals

1. Get comfortable with the notation

2. See how tensor network methods are used in quantum information/condensed matter

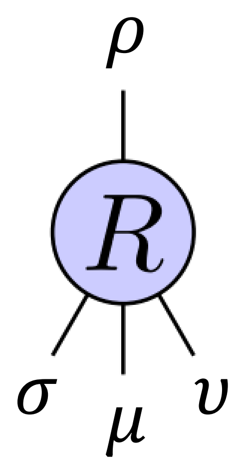
Reference

- These slides use figures from the lecture notes by Bridgeman and Chubb:
    - *Hand-waving and Interpretive Dance: An Introductory Course on Tensor Networks*, J. Phys. A: Math. Theor. 50 223001 (2017) arXiv:1603.03039

- Much of what we will cover can also be found in lectures 1-3 in those notes.

Tensors

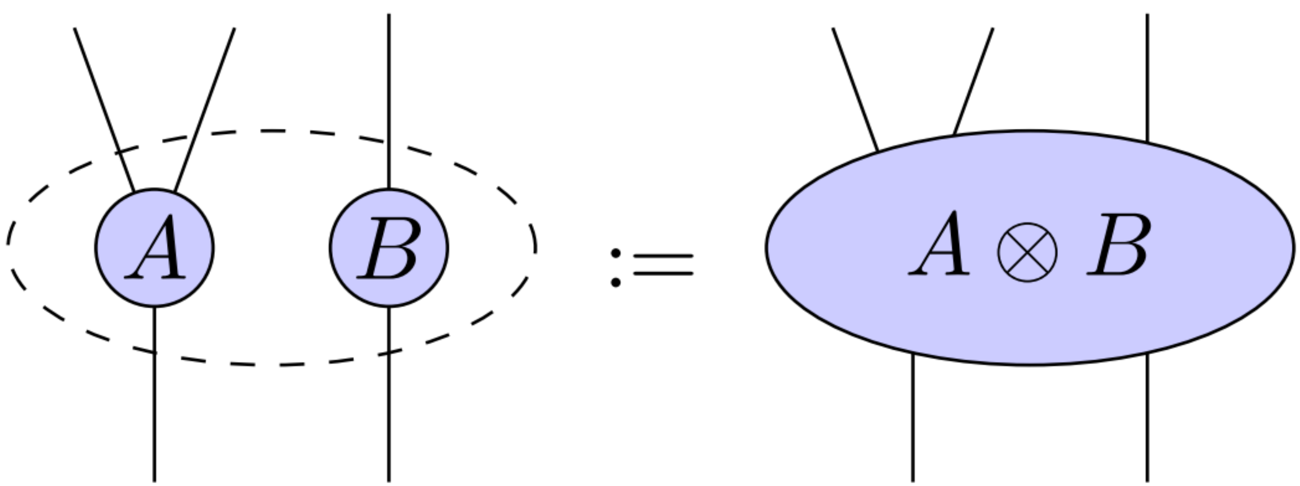
$$R^{\rho}{}_{\sigma\mu\nu} \Longrightarrow$$



- Represent tensors as shapes
- Indices represented as legs

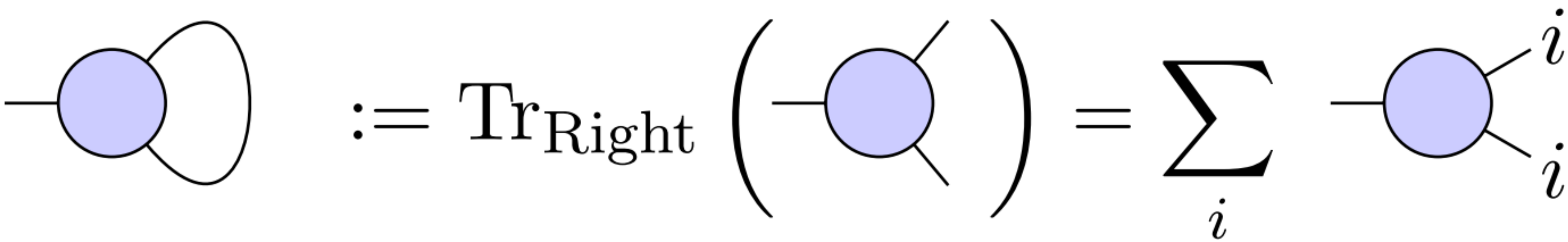
Combining tensors: tensor product

- $[A \otimes B]_{i_1,\ldots i_r; j_1,\ldots j_s} \coloneqq A_{i_1,\ldots i_r} \cdot B_{j_1,\ldots j_s}$

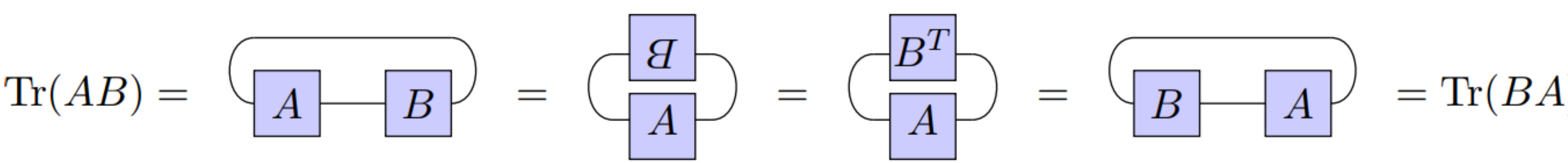


Contracting indices: trace

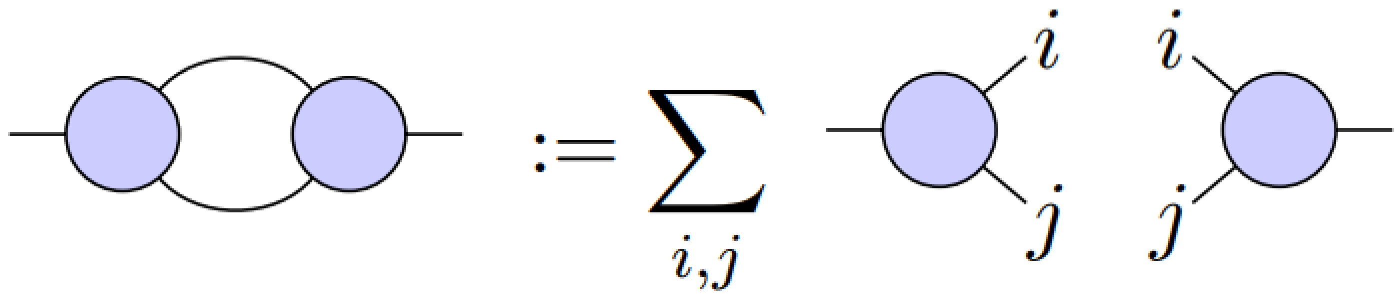
- $\left[\mathrm{Tr}_{x,y}A\right]_{i_1,\ldots i_{x-1},i_{x+1},\ldots i_{y-1}i_{y+1},\ldots i_r} := \sum_{\alpha=1}^{d_x} A_{i_1,\ldots i_{x-1},\alpha,i_{x+1},\ldots i_{y-1},\alpha,i_{y+1},\ldots i_r}$

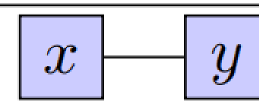
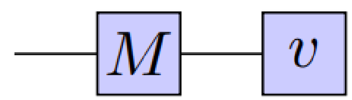
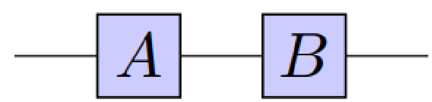
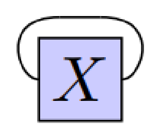


- Cyclic property of the trace:



Contracting indices: generally



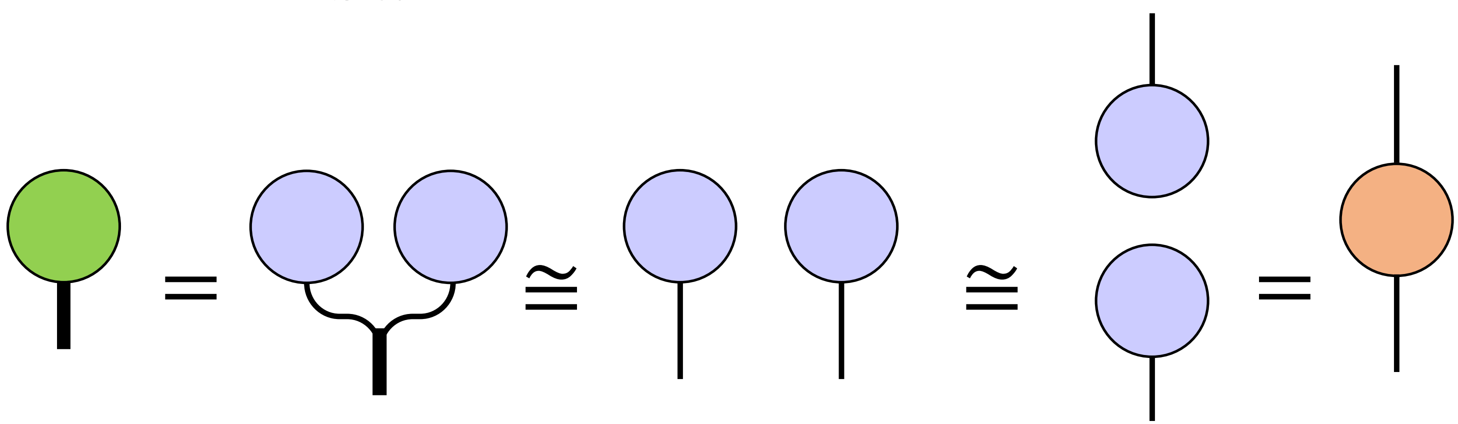
Conventional	Einstein	TNN
$\langle \vec{x}, \vec{y} \rangle$	$x_\alpha y^\alpha$	
$M\vec{v}$	$M^\alpha{}_\beta v^\beta$	
AB	$A^\alpha{}_\beta B^\beta{}_\gamma$	
$\mathrm{Tr}(X)$	$X^\alpha{}_\alpha$	

Grouping and splitting indices

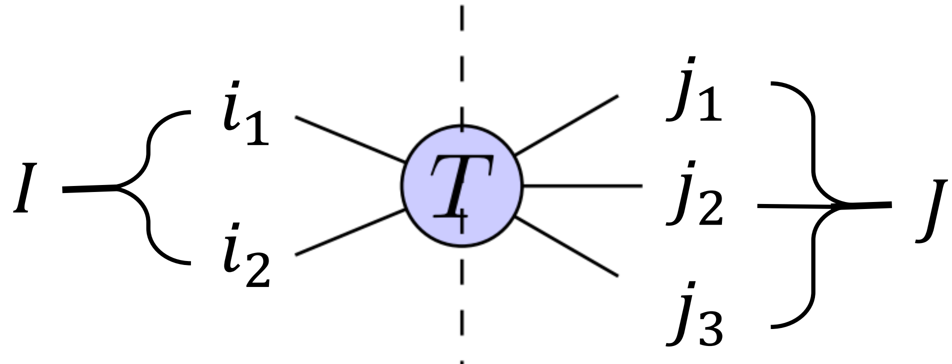
- $\mathbb{C}^{a_1 \times \cdots \times a_n} \cong \mathbb{C}^{b_1 \times \cdots \times b_m} \quad \Leftrightarrow \quad \prod_{i=1}^{n} a_i = \prod_{i=1}^{m} b_i$

- E.g. $\qquad \mathbb{C}^4 \cong \mathbb{C}^{2\times 2} \cong \mathbb{C}^2 \otimes (\mathbb{C}^2)^*$

$$\begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix} \cong \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \cong \begin{pmatrix} ac & ad \\ bc & bd \end{pmatrix}$$

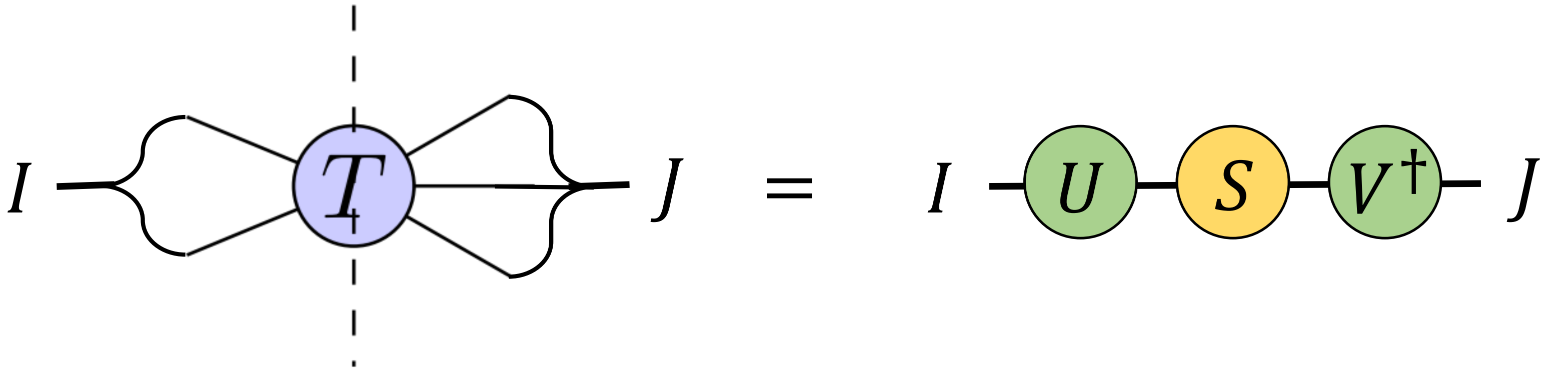


Turning tensors into matrices



- $T_{i_1 \ldots i_n; j_1 \ldots j_m} = T_{I;J}$
- $I = \; i_1 + \dim(i_1)\, i_2 + \dim(i_1)\dim(i_2) i_3 \ldots$
- $J = \; j_1 + \dim(j_1)\, j_2 + \dim(j_1)\dim(j_2) j_3 \ldots$
- E.g., if each index of T has dimension 2
  - $|i_1\rangle \otimes |i_2\rangle \cong |I\rangle$, where i_1 and $i_2 \in \{0, 1\}$, $I \in \{0, 1, 2, 3\}$
  - $\langle j_1| \otimes \langle j_2| \otimes \langle j_3| \cong \langle J|$, where j_1, j_2, and $j_3 \in \{0, 1\}$, $J \in \{0, 1, 2, 3, 4, 5, 6, 7\}$

Matrix decompositions



- Singular value decomposition:
  - $T_{I;J} = \sum_\alpha U_{I;\alpha} S_{\alpha;\alpha} V^*_{J;\alpha}$, where $U^\dagger U = I, S > 0$ is diagonal, $V^\dagger V = I$
- For square matrices: Polar decomposition:
  - $T = WP$, where $W = UV^\dagger$ is unitary, $P = VSV^\dagger > 0$
- For Hermitian matrices: Spectral decomposition
  - $T = UDU^\dagger$, where U is unitary and D is diagonal & real

Isometries vs Unitaries

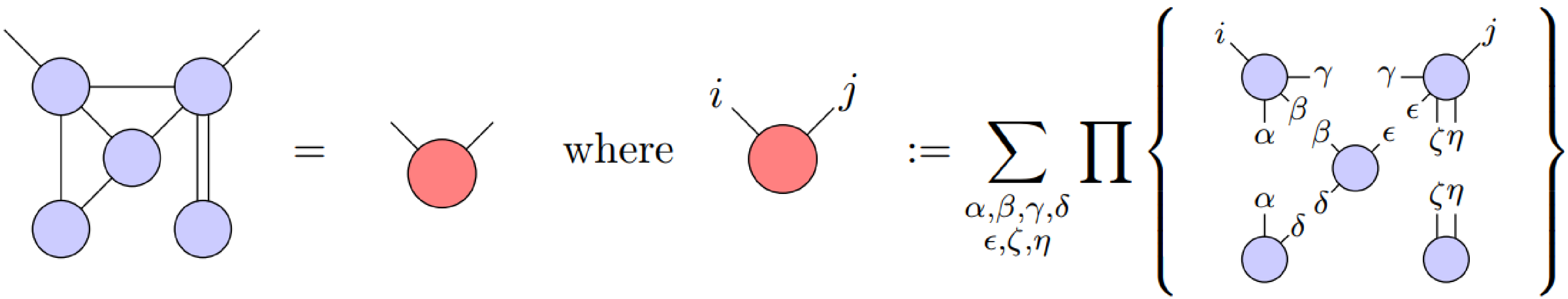
- Unitaries map $\mathbb{C}^d \to \mathbb{C}^d$
  - Written as $U = \sum_{i=1}^{d} |v_i\rangle\langle e_i|$, for some $\mathbb{C}^d$ orthonormal bases $\{|e_i\rangle\}$ and $\{|v_i\rangle\}$
  - Have $U^\dagger U = I = UU^\dagger$ over $\mathbb{C}^d$
- Isometries $\mathbb{C}^k \to \mathbb{C}^d$ for $k < d$
  - Written as $W = \sum_{i=1}^{k} |v_i\rangle\langle e_i|$, for some $\mathbb{C}^k$ orthonormal basis $\{|e_i\rangle\}$, and $\{|v_i\rangle\}$ a strict subset of orthonormal basis elements of $\mathbb{C}^d$
  - Have $W^\dagger W = I$ over $\mathbb{C}^k$
  - Have $WW^\dagger = \Pi_k$, a rank *k* projector over $\mathbb{C}^d$

Rank-*k* approximations to tensors

1. Treat the tensor as a matrix by grouping the legs into two sets (rows and columns)
2. Perform singular value decomposition
3. Set all but the largest *k* singular values equal to zero (known as *k*-trimming)

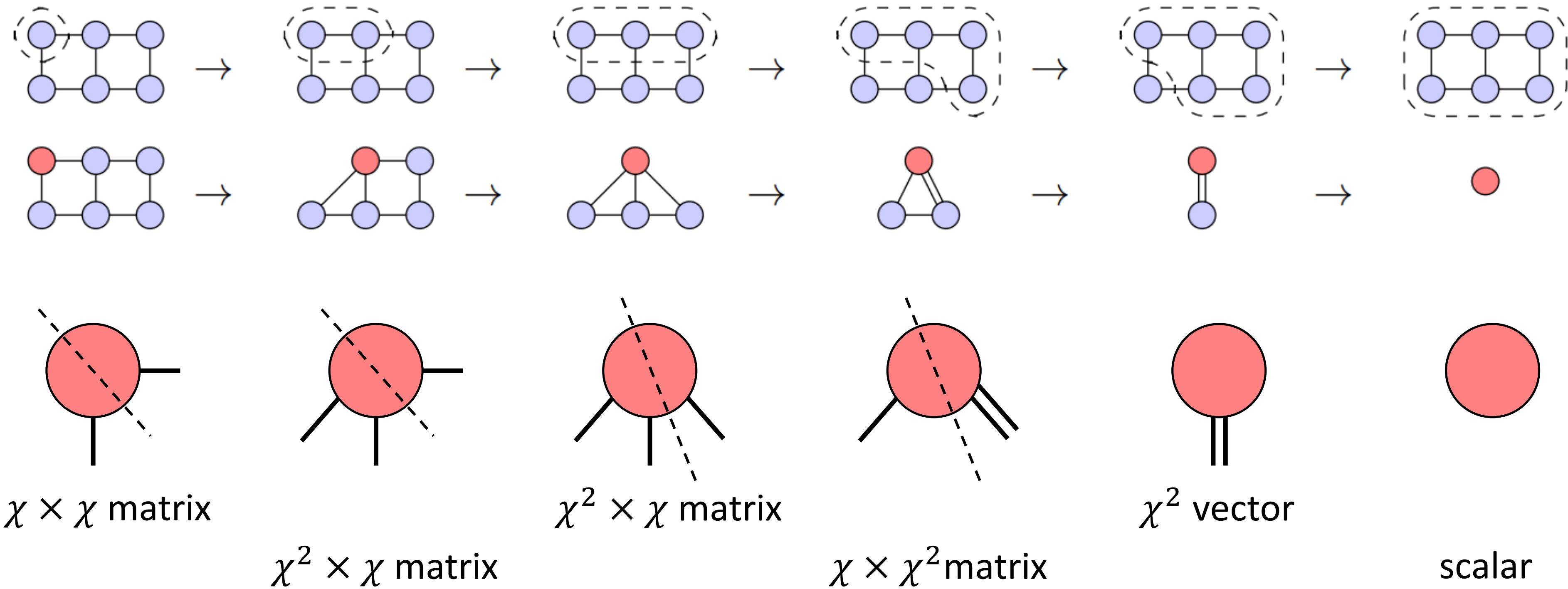
- Eckart-Young theorem:
  - Let $X = USV^\dagger$. Define $X^{(k)} = US^{(k)}V^\dagger$ to be the *k*-trimmed version of S. Then $\left\|X - X^{(k)}\right\|_F \leq \|X - Y\|_F$ for all rank-*k* matrices Y.
  - Note: $\|A\|_F = \sqrt{\mathrm{Tr}(A^{\mathrm{T}}A)}$

Contracting tensor networks



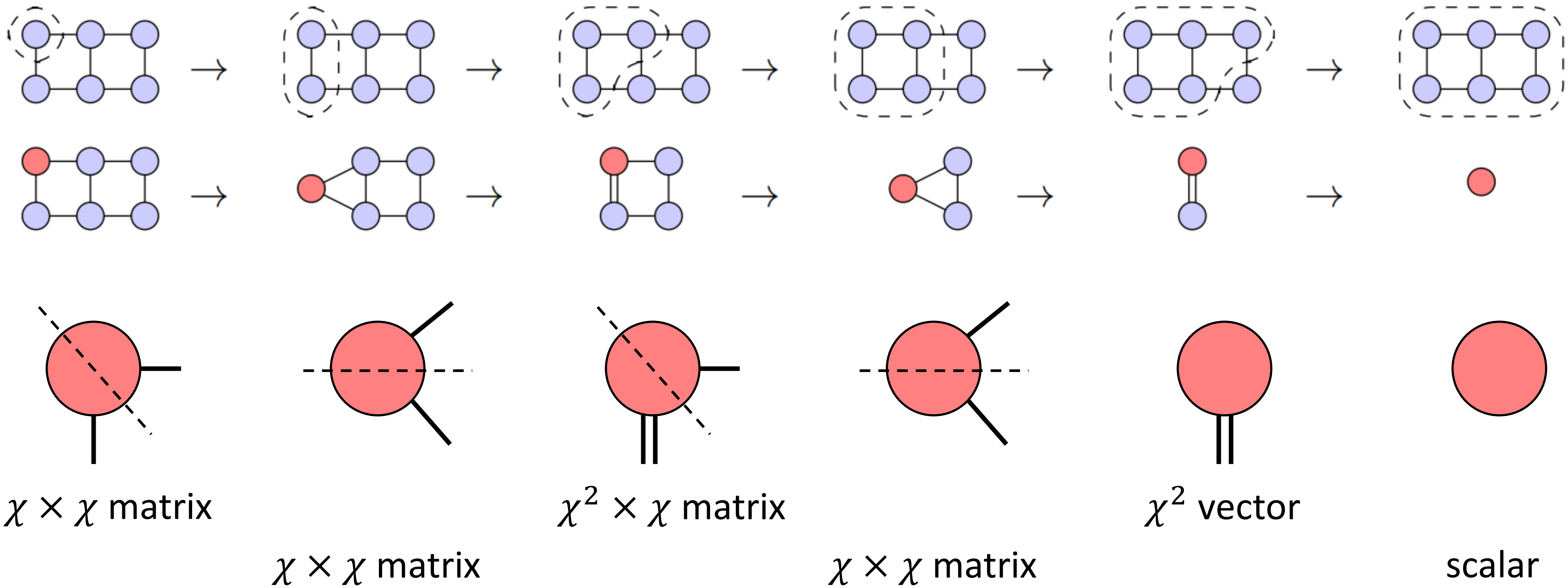
- Contracting general tensor networks is challenging
- Fortunately, we can reduce this to sequential matrix multiplication

Bubbling



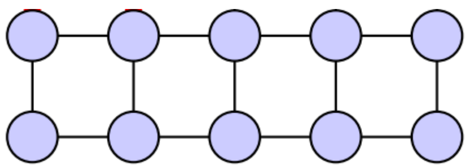
Important note: horizontal-first bubbling on a $2 \times l$ lattice will result in a $\chi^{l-1} \times \chi$ intermediate matrix

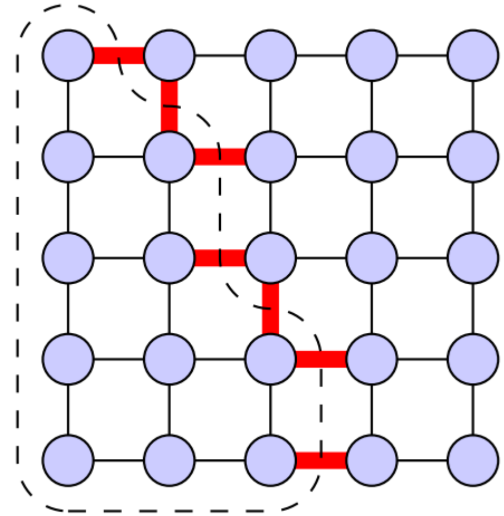
Bubbling: order matters!



Important note: vertical-first bubbling on a $2 \times l$ lattice will result in no intermediate matrices larger than $\chi^2 \times \chi$ or $\chi \times \chi^2$

1D vs 2D networks

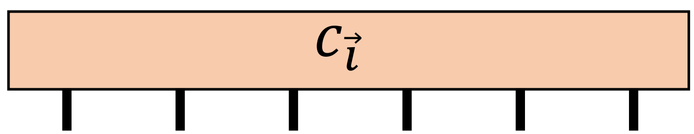




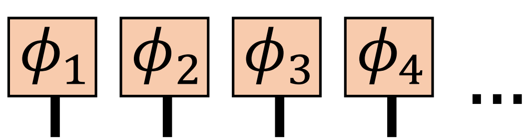
- There is no choice of bubbling that can avoid exponentially growing matrices on the 2D square lattice network.
- Contracting such networks is very hard.
  - Contains #P-complete problems
  - Approximating such contractions is Post-BQP hard

TNN and quantum information

- General quantum states
  - $|\psi\rangle = \sum_{i_1 i_2 \ldots = 0}^{1} c_{i_1 i_2 \ldots} |i_1 i_2 \ldots\rangle$

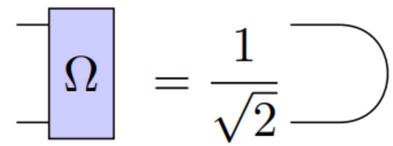
    $=$ 

- Product states
  - $|\psi\rangle = \otimes_{i=1}^{n} |\phi_i\rangle$



TNN and quantum information

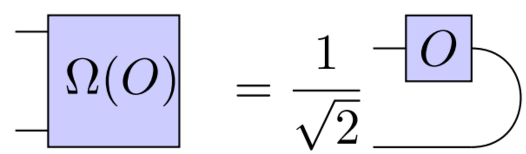
- The Bell basis:
  - $\left|\Phi^{\pm}\right\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \left|\Psi^{\pm}\right\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$



  - Refer to $\left|\Phi^{+}\right\rangle$ as $|\Omega\rangle$

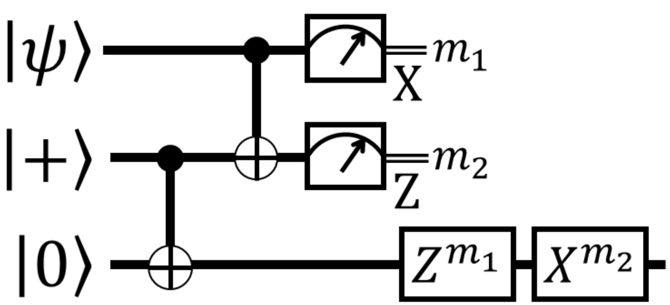
$$|\Omega\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \xrightleftharpoons[\text{Matricise}]{\text{Vectorise}} \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = I/\sqrt{2}.$$

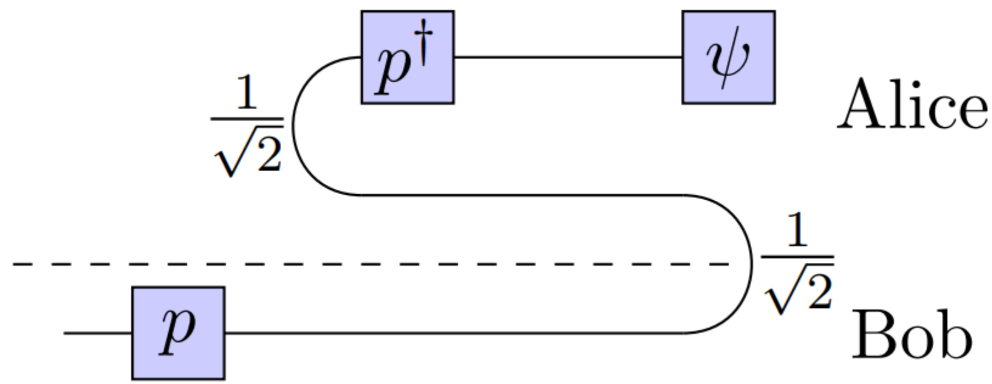
  - $|\Omega(I)\rangle = \left|\Phi^{+}\right\rangle$
  - $|\Omega(X)\rangle = \left|\Psi^{+}\right\rangle$
  - $|\Omega(Y)\rangle \propto |\Psi^{-}\rangle$
  - $|\Omega(Z)\rangle = |\Phi^{-}\rangle$



Teleportation

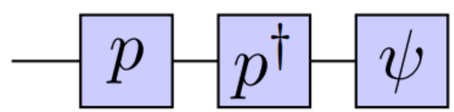
- If Alice and Bob share an entangled resource state, and are allowed to perform classical communication, then Alice can perfectly transmit an arbitrary quantum state to Bob





$|\Omega(I)\rangle = |\Phi^+\rangle$
$|\Omega(X)\rangle = |\Psi^+\rangle$
$|\Omega(Y)\rangle \propto |\Psi^-\rangle$
$|\Omega(Z)\rangle = |\Phi^-\rangle$

$= \frac{1}{2}$

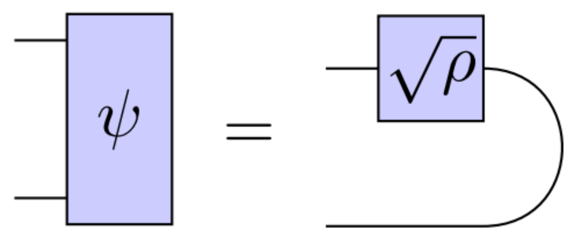


$= |\psi\rangle/2$

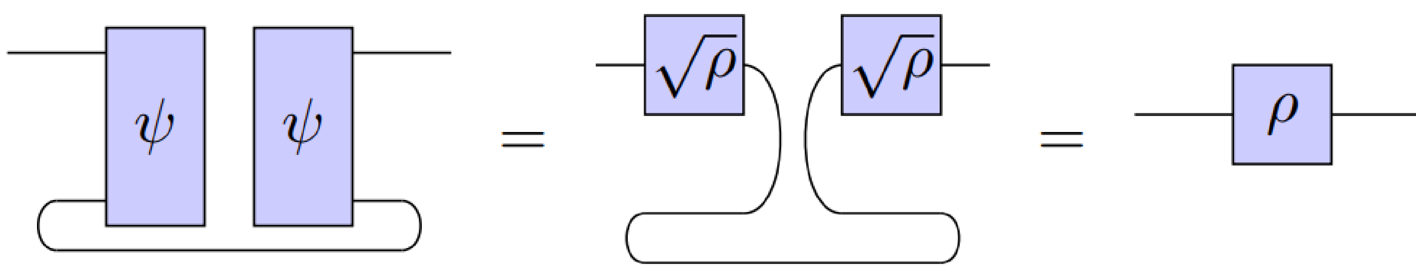
Purification

- Given a mixed state ρ over some Hilbert space $\mathcal{H}_1$, find a pure state $|\psi\rangle$ from an extended Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ s.t. $\rho = \mathrm{Tr}_2(|\psi\rangle\langle\psi|)$

Choose:

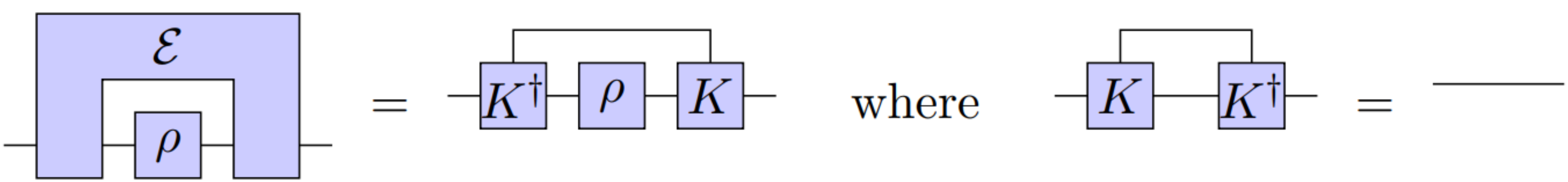


$$\mathrm{Tr}_2\Big(|\psi\rangle\langle\psi|\Big) =$$



Stinespring dilation

- A quantum channel $\mathcal{E}$ that takes valid density matrices to valid density matrices is a completely-positive trace-preserving (CPTP) map.
- In the Kraus operator decomposition:
  - $\mathcal{E}(\rho) = \sum_i K_i^\dagger \rho K_i$, where $\sum_i K_i K_i^\dagger = I$
- In tensor network notation



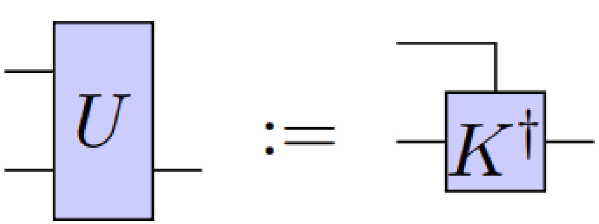
Note the unfortunate dagger convention here

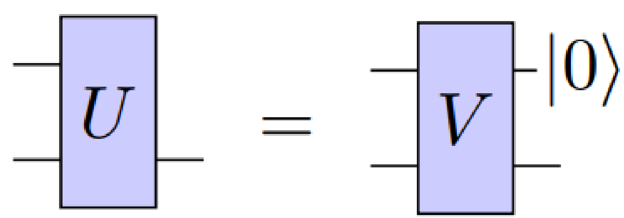
Stinespring dilation

- The Stinespring dilation theorem says that any CPTP map $\mathcal{E}$ can be made by

1. Embedding the system in a larger Hilbert space
2. Evolving the enlarged system under unitary dynamics
3. Reducing back to the original system via partial trace

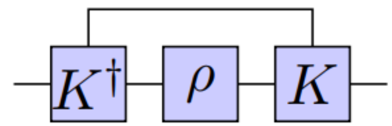
- To see this, we use that the tensor $K_i^\dagger$ is an isometry



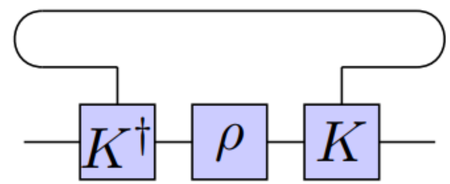


Stinespring dilation

$$\mathcal{E}(\rho) = \sum_i K_i^\dagger \rho K_i =$$



$$=$$



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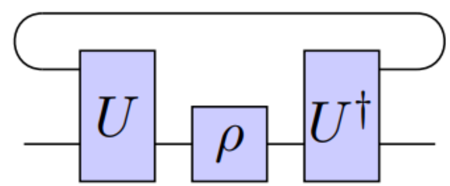

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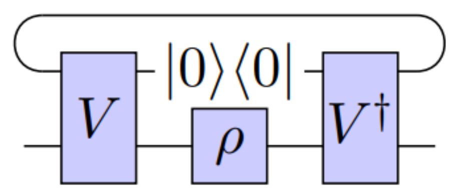
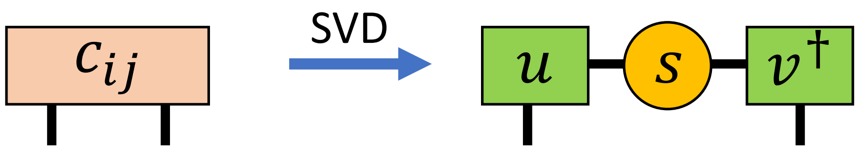


$$= \mathrm{Tr}_1 \left[V^\dagger \left(\rho \otimes |0\rangle\langle 0| \right) V \right]$$

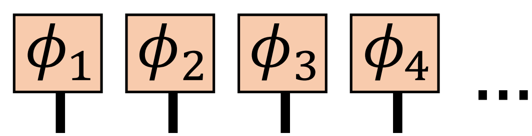
The Schmidt decomposition

- Given a bipartite quantum state
  - $|\psi\rangle_{AB} = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} |i\rangle \otimes |j\rangle$
- We can always write it as
  - $|\psi\rangle_{AB} = \sum_{i=1}^{\chi \leq d} s_i |u(i)\rangle \otimes |v(j)\rangle$
  - χ is known as the Schmidt rank
  - Entanglement entropy $S = -\sum_{i=1}^{\chi} s_i \log s_i \leq \log \chi$

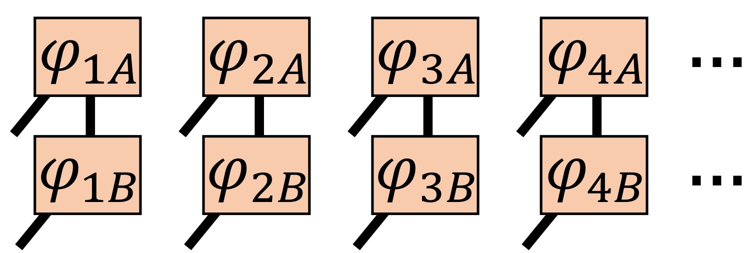


The geometry of entanglement

- Under special circumstances, a many-body quantum system will have *structured* entanglement entropy. We can see this using tensor network notation.
- Example 1: Product state $|\psi\rangle = \otimes_{i=1}^{n} |\phi_i\rangle$

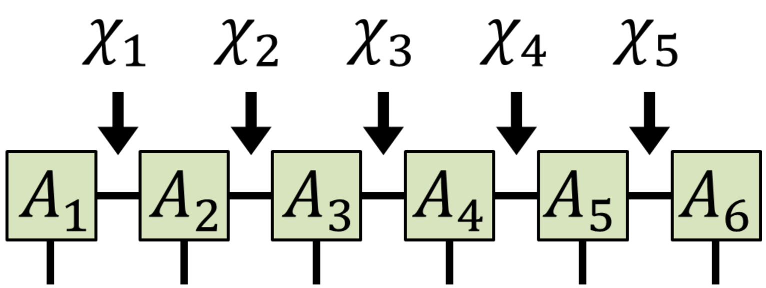


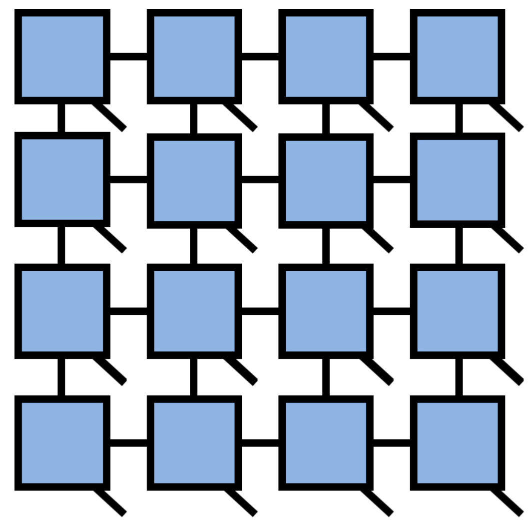
- Example 2: Entangled pairs $|\psi\rangle = \otimes_{i=1}^{n} |\varphi_i\rangle_{AB}$



The geometry of entanglement

- Example 3: *d*-dimensional regular lattices



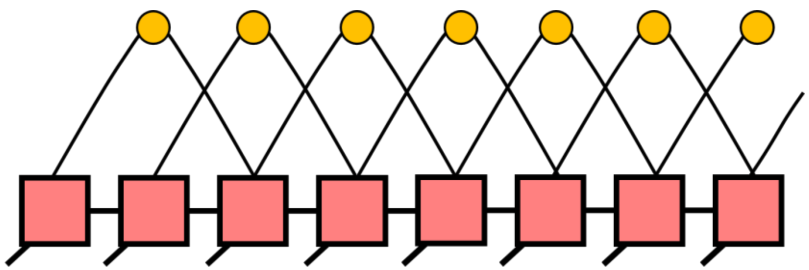


Area law for entanglement $S \sim \log \chi^{d-1}$

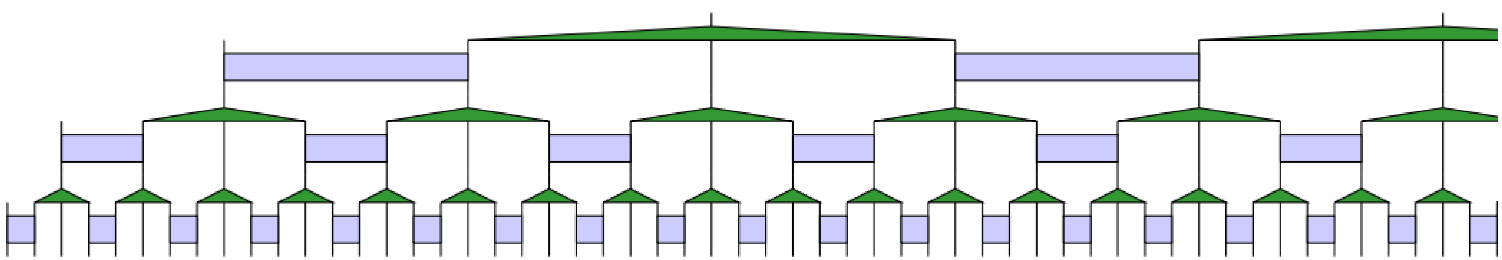
The geometry of entanglement

- Example 4: other lattices

Extra layer gives longer range interactions



Area law for entanglement $S \sim \log L(A)$



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Connection to physics

- Choose a state out of the many-body Hilbert space. Generically, it will have maximal entanglement entropy with respect to any bipartition.
- States that are considered physically reasonable correspond to low-energy sectors of Hamiltonians that are local with respect to a *d*-dimensional lattice and have *unique* ground states (or possibly bounded degeneracy).

Area law conjecture

- Given the unique ground state of a *gapped*, local Hamiltonian on a *d*-dimensional lattice, that state satisfies a boundary law with respect to entanglement entropy.
- Proven in 1D
- Consequences in Tensor networks: We can find a good approximation to the ground state using tensors that look like:

Matrix product state approximations