Tensor networks in quantum information

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What is it?

• A diagrammatic method for representing tensors

\[ R^\rho_{\sigma \mu \nu} \quad \implies \quad R \]

• Why is this useful?
• Gives a convenient way of bookkeeping how much entanglement there is between different parts of a quantum system
• Has resulted in many useful numerical and analytic methods for analysing interesting many-body quantum states.
Goals

1. Get comfortable with the notation

2. See how tensor network methods are used in quantum information/condensed matter
Reference

• These slides use figures from the lecture notes by Bridgeman and Chubb:

• Much of what we will cover can also be found in lectures 1-3 in those notes.
Tensors

- Represent tensors as shapes
- Indices represented as legs

\[ R^\rho_{\sigma \mu \nu} \]
Combining tensors: tensor product

- \([A \otimes B]_{i_1, \ldots, i_r; j_1, \ldots, j_s} := A_{i_1, \ldots, i_r} \cdot B_{j_1, \ldots, j_s}\)
Contracting indices: trace

- \([\text{Tr}_{x,y} A]_{i_1, \ldots, i_{x-1}, i_{x+1}, \ldots, i_{y-1}, i_{y+1}, \ldots, i_r} := \sum_{\alpha=1}^{d_x} A_{i_1, \ldots, i_{x-1}, \alpha, i_{x+1}, \ldots, i_{y-1}, \alpha, i_{y+1}, \ldots, i_r}\)

\[\begin{array}{c}
\text{:= \text{Tr}_{\text{Right}} \left( \begin{array}{c}
\end{array} \right) = \sum_i \begin{array}{c}
\end{array} \right)}
\end{array}\]

- Cyclic property of the trace:

\(\text{Tr}(AB) = \begin{array}{c}
\begin{array}{c}
A \quad B
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
B^T \quad A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
B \quad A
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
B \quad A
\end{array}
\end{array} = \text{Tr}(BA)\)
Contracting indices: generally

\[ \sum_{i,j} \]

<table>
<thead>
<tr>
<th>Conventional</th>
<th>Einstein</th>
<th>TNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle \bar{x}, \bar{y} \rangle )</td>
<td>( x_\alpha y^\alpha )</td>
<td>( x \rightarrow y )</td>
</tr>
<tr>
<td>( M\bar{v} )</td>
<td>( M^{\alpha}_\beta v^\beta )</td>
<td>( M \rightarrow v )</td>
</tr>
<tr>
<td>( AB )</td>
<td>( A^\alpha_\beta B^\beta_\gamma )</td>
<td>( A \rightarrow B )</td>
</tr>
<tr>
<td>( \text{Tr}(X) )</td>
<td>( X^\alpha_\alpha )</td>
<td>( X )</td>
</tr>
</tbody>
</table>
Grouping and splitting indices

• $\mathbb{C}^{a_1 \times \cdots \times a_n} \cong \mathbb{C}^{b_1 \times \cdots \times b_m} \iff \Pi_{i=1}^{n} a_i = \Pi_{i=1}^{m} b_i$

• E.g. $\mathbb{C}^4 \cong \mathbb{C}^{2 \times 2} \cong \mathbb{C}^2 \otimes (\mathbb{C}^2)^*$

$$
\begin{pmatrix}
ac \\
ad \\
bc \\
bd
\end{pmatrix} \cong
\begin{pmatrix}
\begin{pmatrix}
a \\
b
\end{pmatrix} \otimes \begin{pmatrix}
c \\
d
\end{pmatrix}
\cong
\begin{pmatrix}
ac & ad \\
bc & bd
\end{pmatrix}
\end{pmatrix}
$$
Turning tensors into matrices

\[ T_{i_1 \ldots i_n; j_1 \ldots j_m} = T_{I; J} \]

\[ I = i_1 + \text{dim}(i_1) i_2 + \text{dim}(i_1) \text{dim}(i_2) i_3 \ldots \]

\[ J = j_1 + \text{dim}(j_1) j_2 + \text{dim}(j_1) \text{dim}(j_2) j_3 \ldots \]

E.g., if each index of \( T \) has dimension 2

- \( |i_1\rangle \otimes |i_2\rangle \cong |I\rangle \), where \( i_1 \) and \( i_2 \in \{0, 1\} \), \( I \in \{0, 1, 2, 3\} \)
- \( \langle j_1| \otimes \langle j_2| \otimes \langle j_3| \cong \langle J| \), where \( j_1, j_2, \) and \( j_3 \in \{0, 1\} \), \( J \in \{0, 1, 2, 3, 4, 5, 6, 7\} \)
Matrix decompositions

- Singular value decomposition:
  - $T_{I;J} = \sum_\alpha U_{I;\alpha} S_{\alpha;\alpha} V_{J;\alpha}^*$, where $U^\dagger U = I$, $S > 0$ is diagonal, $V^\dagger V = I$

- For square matrices: Polar decomposition:
  - $T = WP$, where $W = UV^\dagger$ is unitary, $P = VSV^\dagger > 0$

- For Hermitian matrices: Spectral decomposition
  - $T = UDU^\dagger$, where $U$ is unitary and $D$ is diagonal & real
Isometries vs Unitaries

• Unitaries map $\mathbb{C}^d \rightarrow \mathbb{C}^d$
  • Written as $U = \sum_{i=1}^{d} |v_i\rangle \langle e_i|$, for some $\mathbb{C}^d$ orthonormal bases $\{|e_i\rangle\}$ and $\{|v_i\rangle\}$
  • Have $U^\dagger U = I = UU^\dagger$ over $\mathbb{C}^d$

• Isometries $\mathbb{C}^k \rightarrow \mathbb{C}^d$ for $k < d$
  • Written as $W = \sum_{i=1}^{k} |v_i\rangle \langle e_i|$, for some $\mathbb{C}^k$ orthonormal basis $\{|e_i\rangle\}$, and $\{|v_i\rangle\}$ a strict subset of orthonormal basis elements of $\mathbb{C}^d$
  • Have $W^\dagger W = I$ over $\mathbb{C}^k$
  • Have $WW^\dagger = \Pi_k$, a rank $k$ projector over $\mathbb{C}^d$
Rank-\(k\) approximations to tensors

1. Treat the tensor as a matrix by grouping the legs into two sets (rows and columns)
2. Perform singular value decomposition
3. Set all but the largest \(k\) singular values equal to zero (known as \(k\)-trimming)

- Eckart-Young theorem:
  - Let \(X = USV^\dagger\). Define \(X^{(k)} = US^{(k)}V^\dagger\) to be the \(k\)-trimmed version of \(S\). Then \(\|X - X^{(k)}\|_F \leq \|X - Y\|_F\) for all rank-\(k\) matrices \(Y\).
  - Note: \(\|A\|_F = \sqrt{\text{Tr}(A^TA)}\)
Contracting tensor networks

Contracting general tensor networks is challenging
Fortunately, we can reduce this to sequential matrix multiplication
Bubbling

Important note: horizontal-first bubbling on a $2 \times l$ lattice will result in a $\chi^{l-1} \times \chi$ intermediate matrix
Important note: vertical-first bubbling on a $2 \times l$ lattice will result in no intermediate matrices larger than $\chi^2 \times \chi$ or $\chi \times \chi^2$
1D vs 2D networks

• There is no choice of bubbling that can avoid exponentially growing matrices on the 2D square lattice network.
• Contracting such networks is very hard.
  • Contains \#P-complete problems
  • Approximating such contractions is Post-BQP hard
TNN and quantum information

• General quantum states
  • \( |\psi\rangle = \sum_{i_1i_2...=0}^1 c_{i_1i_2...} |i_1i_2...\rangle \)
  
\[
= \begin{array}{c}
  c_i \\
\end{array}
\]

• Product states
  • \( |\psi\rangle = \bigotimes_{i=1}^n |\phi_i\rangle \)

\[
\begin{array}{cccc}
  \phi_1 & \phi_2 & \phi_3 & \phi_4 \\
\end{array}
\]

\[\ldots\]
TNN and quantum information

- The Bell basis:
  - $|\Phi^\pm\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}$, $|\Psi^\pm\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}$
  - Refer to $|\Phi^+\rangle$ as $|\Omega\rangle$

\[
|\Omega\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \xrightarrow{\text{Vectorise/Matricise}} \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I/\sqrt{2}.
\]

- $|\Omega(I)\rangle = |\Phi^+\rangle$
- $|\Omega(X)\rangle = |\Psi^+\rangle$
- $|\Omega(Y)\rangle \propto |\Psi^-\rangle$
- $|\Omega(Z)\rangle = |\Phi^-\rangle$

\[
\Omega = \frac{1}{\sqrt{2}}
\]

\[
\Omega(O) = \frac{1}{\sqrt{2}}
\]
Teleportation

- If Alice and Bob share an entangled resource state, and are allowed to perform classical communication, then Alice can perfectly transmit an arbitrary quantum state to Bob.

\[
|\psi\rangle = |\Phi^+\rangle
|\Omega(X)\rangle = |\Psi^+\rangle
|\Omega(Y)\rangle \propto |\Psi^-\rangle
|\Omega(Z)\rangle = |\Phi^-\rangle
\]

\[
= \frac{1}{2} \frac{1}{\sqrt{2}} p p^\dagger \psi = |\psi\rangle / 2
\]
Purification

• Given a mixed state $\rho$ over some Hilbert space $\mathcal{H}_1$, find a pure state $|\psi\rangle$ from an extended Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ s.t. $\rho = \text{Tr}_2(|\psi\rangle\langle\psi|)$

Choose:

$$\text{Tr}_2(|\psi\rangle\langle\psi|) = \psi \sqrt{\rho} = \sqrt{\rho} \sqrt{\rho} = \rho$$
Stinespring dilation

• A quantum channel $\mathcal{E}$ that takes valid density matrices to valid density matrices is a completely-positive trace-preserving (CPTP) map.

• In the Kraus operator decomposition:
  - $\mathcal{E}(\rho) = \sum_i K_i^\dagger \rho K_i$, where $\sum_i K_i K_i^\dagger = I$
  - In tensor network notation

Note the unfortunate dagger convention here
Stinespring dilation

• The Stinespring dilation theorem says that any CPTP map $\mathcal{E}$ can be made by
  1. Embedding the system in a larger Hilbert space
  2. Evolving the enlarged system under unitary dynamics
  3. Reducing back to the original system via partial trace

• To see this, we use that the tensor $K_i^\dagger$ is an isometry

\[
U := K^\dagger = U V |0\rangle
\]
Stinespring dilation

\[ \mathcal{E}(\rho) = \sum_i K_i^\dagger \rho K_i = \begin{array}{c}
\begin{array}{c}
K^\dagger \quad \rho \quad K
\end{array}
\end{array} \]

\[ = \begin{array}{c}
\begin{array}{c}
K^\dagger \quad \rho \quad K
\end{array}
\end{array} \]

\[ = \begin{array}{c}
\begin{array}{c}
U \quad \rho \quad U^\dagger
\end{array}
\end{array} \]

\[ = \begin{array}{c}
\begin{array}{c}
V \quad |0\rangle \langle 0| \quad V^\dagger
\end{array}
\end{array} \]

\[ = \text{Tr}_1 \left[ V^\dagger (\rho \otimes |0\rangle \langle 0|) V \right] \]
The Schmidt decomposition

• Given a bipartite quantum state
  - $|\psi\rangle_{AB} = \sum_{i=1}^{d} \sum_{j=1}^{d} c_{ij} |i\rangle \otimes |j\rangle$

• We can always write it as
  - $|\psi\rangle_{AB} = \sum_{i=1}^{\chi \leq d} s_i |u(i)\rangle \otimes |v(j)\rangle$
  - $\chi$ is known as the Schmidt rank
  - Entanglement entropy $S = - \sum_{i=1}^{\chi} s_i \log s_i \leq \log \chi$
The geometry of entanglement

• Under special circumstances, a many-body quantum system will have structured entanglement entropy. We can see this using tensor network notation.

• Example 1: Product state $|\psi\rangle = \bigotimes_{i=1}^{n} |\phi_i\rangle$

• Example 2: Entangled pairs $|\psi\rangle = \bigotimes_{i=1}^{n} |\varphi_i\rangle_{AB}$
The geometry of entanglement

- Example 3: $d$-dimensional regular lattices

Area law for entanglement $S \sim \log \chi^{d-1}$
The geometry of entanglement

• Example 4: other lattices

Extra layer gives longer range interactions

Area law for entanglement $S \sim \log L(A)$
Connection to physics

• Choose a state out of the many-body Hilbert space. Generically, it will have maximal entanglement entropy with respect to any bipartition.

• States that are considered physically reasonable correspond to low-energy sectors of Hamiltonians that are local with respect to a $d$-dimensional lattice and have unique ground states (or possibly bounded degeneracy).
Area law conjecture

• Given the unique ground state of a gapped, local Hamiltonian on a $d$-dimensional lattice, that state satisfies a boundary law with respect to entanglement entropy.

• Proven in 1D

• Consequences in Tensor networks: We can find a good approximation to the ground state using tensors that look like:
Matrix product state approximations

\[ \psi = L \lambda R \]

\[ \psi = M^{(1)} \lambda^{(1)} R^{(1)} \]

\[ = M^{(1)} \lambda^{(1)} M^{(2)} \lambda^{(2)} R^{(2)} \]

\[ = M^{(1)} \lambda^{(1)} M^{(2)} \lambda^{(2)} M^{(3)} \lambda^{(3)} M^{(4)} \]

\[ |\psi\rangle = A^{(1)} A^{(2)} A^{(3)} A^{(4)} \]