Physics 572: Homework #1 Solutions

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Problem 1: Entanglement and correlations.

Part I (10 points): Since $p(a|b) = p(a|b')$ for all a, b, b', we can fix some b_0 such that $p(a|b) = p(a|b_0)$ for all a, b, and so

$$p(a) = \sum_b p(a,b) = \sum_b p(b)p(a|b) = p(a|b_0)\sum_b p(b) = p(a|b_0) \tag{1}$$

Therefore $(\forall a, b)p(a) = p(a|b) = p(a,b)/p(b) \implies (\forall a, b)\ p(a,b) = p(a)p(b)$. Now $p(b|a) = p(a,b)/p(a) = p(b)$, for all a, b, and so $p(b|a) = p(b) = p(b|a')$ for all a, a', b as claimed.

Part II (10 points): If the measurement results of Alice and Bob are uncorrelated then $p(a, b) = p(b)p(b)$,

$$\rho_{AB} = \sum_{a,b} p(a,b)(|a\rangle\langle a| \otimes |b\rangle\langle b|) = \sum_{a,b} p(a)p(b)\,(|a\rangle\langle a| \otimes |b\rangle\langle b|) \tag{2}$$

$$= \left(\sum_a p(a)|a\rangle\langle a|\right) \otimes \left(\sum_b p(b)|b\rangle\langle b|\right) = \rho_A \otimes \rho_B \tag{3}$$

If the measurements results are correlated, $p(a,b) \neq p(a)p(b)$, then we have $\rho_{AB} \neq \rho_A \otimes \rho_B$. But

$$\rho_{AB} = \sum_{a,b} p(a,b)(|a\rangle\langle a| \otimes |b\rangle\langle b|)$$

is a separable state (a mixture of of unentangled states). States ρ_{AB} of this form are never entangled, even if the measurement results are correlated (separable states represent systems with classical correlations).

Part III (30 points): The implication is true in one direction: if the pure state $|\psi_{AB}\rangle$ is unentangled, then the measurement results in any choice of tensor product basis will be uncorrelated. To see this,

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \left(\sum_a \lambda_a|a\rangle\right) \otimes \left(\sum_b \gamma_b|b\rangle\right) = \sum_{a,b} \lambda_a\gamma_b|ab\rangle, \tag{4}$$

and so $p(a,b) = |\lambda_a\gamma_b|^2 = |\lambda_a|^2|\gamma_b|^2 = p(a)p(b)$ implies the measurement results are uncorrelated. However, the other direction of the implication is false: even if a pure state is entangled, there may be some choice of measurement basis which result in uncorrelated outcomes. This can be shown by example,

$$|\psi\rangle = |00\rangle + |10\rangle + |01\rangle - |11\rangle \tag{5}$$

which is an entangled state the arises from applying a controlled Z gate to the state $|++\rangle$. Although this state is entangled, the measurement outcomes in the computational basis are uncorrelated:

$$p(a,b) = 1/4 \quad , \quad p(a) = 1/2, \quad , \quad p(b) = 1/2 \quad a, b \in \{0, 1\}. \tag{6}$$

Problem 2: A distance measure between quantum states.

Part I (10 points): Following the hint we bipartition the event space into $\Omega = A_1 \cup A_2$, defined by

$$A_1 = \{x \in \Omega : p_x \geq \tilde{p}_x\} \quad , \quad A_2 = \{x \in \Omega : p_x < \tilde{p}_x\}. \tag{7}$$

Notice that $|p(A) - \tilde{p}(A)| = \left|\sum_{x\in A}(p_x - \tilde{p}_x)\right|$, and so the maximum will be achieved when all the terms in the sum are nonnegative ($A = A_1$). Furthermore, these subsets satisfy $p(A_1) + p(A_2) = 1$ and $\tilde{p}(A_1) + \tilde{p}(A_2) = 1$, so $|p(A_1) - \tilde{p}(A_1)| = |p(A_2) - \tilde{p}(A_2)|$ and

$$d(p, \tilde{p}) = \frac{1}{2}\sum_x |p_x - \tilde{p}_x| = \frac{1}{2}\left(|p(A_1) - \tilde{p}(A_1)| + |p(A_2) - \tilde{p}(A_2)|\right) = |p(A_1) - \tilde{p}(A_1)| = \max_{A\subseteq\Omega} |p(A) - \tilde{p}(A)|. \tag{8}$$

Part II (20 points): Choosing an arbitrary basis $\{|\psi_a\rangle\}_{a=1}^{|\Omega|}$, the measurement distributions are

$$p_a = \langle\psi_a|\rho|\psi_a\rangle \quad , \quad \tilde{p}_a = \langle\psi_a|\tilde{\rho}|\psi_a\rangle, \tag{9}$$

and the total variation distance is $d(p, \tilde{p}) = \frac{1}{2}\sum_a |p_a - \tilde{p}_a|$. If we diagonalize $\rho - \tilde{\rho} = \sum_i \lambda_i |i\rangle\langle i|$ then

$$\sum_a |p_a - \tilde{p}_a| = \sum_a |\langle\psi_a|\rho - \tilde{\rho}|\psi_a\rangle| = \sum_a \left|\sum_i \lambda_i P_{ia}\right| \leq \sum_a \left(\sum_i |\lambda_i| P_{ia}\right) = \sum_i |\lambda_i| \tag{10}$$

where $P_{ia} = |\langle\psi_a|i\rangle|^2$ is a stochastic matrix (so $P_{ia} \geq 0$ and $\sum_a P_{ia} = 1$ for all i).

Part III (20 points): Choose the operator A to be:

$$A = \frac{1}{2}(\rho - \tilde{\rho}) = \sum_i \frac{\lambda_i}{2}|i\rangle\langle i| \tag{11}$$

so that

$$\mathrm{tr}\left(\sqrt{A^\dagger A}\right) = \mathrm{tr}\left(\sqrt{\sum_i \frac{|\lambda_i|^2}{4}|i\rangle\langle i|}\right) = \sum_i \frac{|\lambda_i|}{2} \tag{12}$$

For any projector $P = \sum_k |\psi_k\rangle\langle\psi_k|$ we have

$$\mathrm{tr}(P(\rho - \tilde{\rho})) = \sum_{i,k} \lambda_i |\langle\psi_k|i\rangle|^2 \leq \sum_{i:\lambda_i\geq 0}\sum_k \lambda_i |\langle\psi_k|i\rangle|^2 \leq \sum_{i:\lambda_i\geq 0} \lambda_i = \frac{1}{2}\sum_i |\lambda_i| \tag{13}$$

where the last step follows because $\sum_i \lambda_i = 0$ (since $\mathrm{tr}(\rho - \tilde{\rho}) = \mathrm{tr}(\rho) - \mathrm{tr}(\tilde{\rho})$). The inequality is saturated when $P = \sum_{k:\lambda_k\geq 0} |k\rangle\langle k|$, and so

$$d(\rho, \tilde{\rho}) = \max_P \mathrm{tr}(P(\rho - \tilde{\rho})). \tag{14}$$

where the max is over projectors of the form $P = \sum_k |\psi_k\rangle\langle\psi_k|$.

Quantum Channels

Part I (10 points): Given any state $|\psi\rangle$, find a complete set of basis: $\{|\phi_0\rangle, |\phi_1\rangle, |\phi_2\rangle, \cdots\}$, with $|\phi_0\rangle = |\psi\rangle$. Choose the Kraus operators $E_k = |\phi_0\rangle\langle\phi_k|$, which satisfy

$$\sum_k E_k^\dagger E_k = \sum_k |\phi_k\rangle\langle\phi_k| = I. \tag{15}$$

Computing the action of the channel on an arbitrary state ρ,

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k |\phi_0\rangle\langle\phi_k|\rho|\phi_k\rangle\langle\phi_0| = |\psi\rangle\langle\psi| \cdot \mathrm{tr}(\rho) = |\psi\rangle\langle\psi|. \tag{16}$$

Part II (10 points): Choose the Kraus operators:

$$E_{jk} = 2^{-n/2}|j\rangle\langle k| \tag{17}$$

where j, k range over complete sets of basis states. The normalization condition is satisfied,

$$\sum_{jk} E_{jk}^\dagger E_{jk} = \sum_{jk} \frac{|k\rangle\langle j|j\rangle\langle k|}{2^n} = I$$

and the action of the channel is

$$\mathcal{E}(\rho) = \sum_{jk} |j\rangle\langle k|\rho|k\rangle\langle j|/2^n = \mathrm{tr}(\rho)\sum_j \frac{|j\rangle\langle j|}{2^n} = \frac{I}{2^n}.$$

To relate this result to the single qubit depolarization channel we discussed in class, one could note that the set of all tensor products of Pauli matrices forms a basis (under complex linear combinations) for the space of complex matrices. Alternatively, using the results from part IV of this problem one could take the partial trace down to a system with a single qubit, then apply the single qubit depolarizing channel from class, and then use the results of part V to tack on a maximally mixed subsystem. By the results of part III the decomposition of these three channels would also be a channel.

Part III (10 points): The Kraus operators are: $\{E_t' E_k\}$, since

$$\sum_{k,t}^{T,S} (E_t' E_k)^\dagger (E_t' E_k) = \sum_t^T E_t'^\dagger \left(\sum_k^S E_k^\dagger E_k\right) E_t' = \sum_t^T E_t'^\dagger E_t' = I \tag{18}$$

and

$$\mathcal{E}' \circ \mathcal{E}(\rho) = \sum_{k,t}^{T,S} E_t'(E_k \rho E_k^\dagger) E_t'^\dagger = \sum_{k,t}^{T,S} (E_t' E_k)\rho(E_t' E_k)^\dagger \tag{19}$$

Part IV (10 points): Beginning from the Schmidt decomposition $|\phi_{AB}\rangle = \sum_k \lambda_k |a_k\rangle \otimes |b_k\rangle$, the reduced state is $\rho_A = \sum_k \lambda_k^2 |a_k\rangle\langle a_k|$. Define the Kraus operators $E_k = |b_k\rangle\langle b_k|$, which satisfy normalization

$$\sum_k E_k E_k^\dagger = \sum_k |b_k\rangle\langle b_k|b_k\rangle\langle b_k| = I$$

and act on ρ as follows,

$$\begin{aligned}\mathcal{E}(\rho) &= \sum_k |b_k\rangle\langle b_k| \left(\sum_{ij} \lambda_i \lambda_j |a_i\rangle\langle j| \otimes |b_i\rangle\langle b_j|\right) |b_k\rangle\langle b_k| \\ &= \sum_{ijk} \lambda_i \lambda_j |a_i\rangle\langle a_j| \otimes \langle b_k|b_i\rangle\langle b_j|b_k\rangle = \sum_k \lambda_k^2 |a_k\rangle\langle a_k| = \mathrm{tr}_B \rho_{AB}.\end{aligned} \tag{20}$$

Part V (10 points): Diagonalize $\rho_B = \sum_k p_k |b_k\rangle\langle b_k|$, the Kraus operators $\{E_k\}$ that set ρ_A to $\rho_A \otimes \rho_B$ are $E_k = I_A \otimes \sqrt{p_k}|b_k\rangle$. Testing normalization,

$$\sum_k E_k^\dagger E_k = \sum_k I_A \otimes p_k \langle b_k|b_k\rangle = I_A \cdot \sum_k p_k = I_A$$

and the action of the channel is

$$\mathcal{E}(\rho_A) = \rho_A \otimes \sum_k p_k |b_k\rangle\langle b_k| = \rho_A \otimes \rho_B.$$

Part VI (20 points)

(a): The Kraus operator sum representation is,

$$\mathcal{E}(\rho) = (1-p)\mathbb{1}\rho\mathbb{1} + pX\rho X. \tag{21}$$

Where X is the Pauli matrix σ_x. Hence the Kraus operators are, $\{\sqrt{1-p}\mathbb{1}, \sqrt{p}X\}$.
(b): The state $|\psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ can be expressed as a density matrix $|\psi\rangle\langle\psi|$ in terms of pauli matrices i.e. a block vector representation.

$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(\mathbb{1} - \cos(2\theta)Z + \sin(2\theta)X\right) \tag{22}$$

After one application of the channel this becomes,

$$\begin{aligned} \mathcal{E}(|\psi\rangle\langle\psi|) &= \begin{bmatrix} \frac{1}{2} + \frac{2p-1}{2}\cos(2\theta) & \frac{1}{2}\sin(2\theta) \\ \frac{1}{2}\sin(2\theta) & \frac{1}{2} - \frac{2p-1}{2}\cos(2\theta) \end{bmatrix} && (23) \\ &= \frac{1}{2}\left(\mathbb{1} - (1-2p)\cos(2\theta)Z + \sin(2\theta)X\right). && (24) \end{aligned}$$

(c): We wish to find a unitary U such that $\mathcal{E}(\rho) = \mathrm{Tr}_E(U\ \rho \otimes |0\rangle\langle 0|\ U^\dagger)$. The following evolution

$$|0\rangle|0\rangle_E \mapsto \sqrt{1-p}|0\rangle|0\rangle_E + \sqrt{p}|1\rangle|1\rangle_E \tag{25}$$

$$|1\rangle|0\rangle_E \mapsto \sqrt{1-p}|1\rangle|0\rangle_E + \sqrt{p}|0\rangle|1\rangle_E \tag{26}$$

is reversible and therefore unitary (we assume it acts as the identity on $|0\rangle|1\rangle_E, |1\rangle|1\rangle_E$), and satisfies

$$\begin{aligned} \langle 0|U|0\rangle_E &= \sqrt{1-p}\mathbb{1} && (27) \\ \langle 1|U|0\rangle_E &= \sqrt{p}X. && (28) \end{aligned}$$

The full unitary can be written,

$$U = \sqrt{1-p}\mathbb{1} \otimes |0\rangle\langle 0| + \sqrt{p}X \otimes |1\rangle\langle 0| + \sqrt{p}X \otimes |0\rangle\langle 1| + \sqrt{1-p}\mathbb{1} \otimes |1\rangle\langle 1|. \tag{29}$$

Note that this is a permutation matrix.

Problem 4: Noisy classical channel.

Given probability distribution as follows:

$$X : p(0) = \frac{1}{2}, \quad p(1) = \frac{1}{2}; \tag{30}$$

$$Y : p(1|1) = 1-\epsilon, \quad p(0|1) = \epsilon, \quad p(1|0) = \epsilon, \quad p(0|0) = 1-\epsilon. \tag{31}$$

Part I (5 points): Entropy of the source: $S(X) = -\sum_i p_i \log p_i = \log 2 = 1$.

Part II (5 points): Using $p(y) = \sum_x p(y|x)p(x)$, we compute

$$p(y=0) = \frac{\epsilon}{2} + \frac{1-\epsilon}{2} = \frac{1}{2} \quad , \quad p(y=1) = \frac{\epsilon}{2} + \frac{1-\epsilon}{2} = \frac{1}{2}$$

and so $S(Y) = 1$.

Part III (5 points): The joint distribution is:

$$p(x=0, y=0) = \frac{1-\epsilon}{2}, \quad p(x=0, y=1) = \frac{\epsilon}{2}; \tag{32}$$

$$p(x=1, y=0) = \frac{\epsilon}{2}, \quad p(x=1, y=1) = \frac{1-\epsilon}{2}. \tag{33}$$

and from this we calculate

$$S(X,Y) = 1 - (1-\epsilon)\log(1-\epsilon) - \epsilon \log \epsilon \tag{34}$$

Part IV (5 points):

From the definition of mutual information,

$$I(X:Y) = S(X) + S(Y) - S(X,Y) = 1 + (1-\epsilon)\log(1-\epsilon) + \epsilon \log \epsilon. \tag{35}$$

Part V (5 points): When $\epsilon = 0, 1$, the value of mutual information is maximized, $I(X:Y) = 1$. This condition means Y is completely determined by X, and so the channel is working as intended, or exactly opposite as intended (and if the latter behavior is known it can be accounted for to use the channel to transmit information).

Part VI (5 points):

When $\epsilon = 1/2$, the value of mutual information is minimal, $I(X:Y) = 0$. This condition means that the source and receiver are completely uncorrelated, and so no transmission of information is possible.