Physics 572: Homework #1

Due Date: March 7$^{\text{th}}$, 2019

Problem 1: Entanglement and correlations. Supposes Alice observes events from the set $\{1, ..., d_A\}$ and Bob observes events from the set $\{1, ..., d_B\}$. The probability that Alice observes a and Bob observes b is $p(a, b)$. The probability that Alice observes a is

$$p(a) = \sum_{b=1}^{d_B} p(a, b).$$

The conditional probability for Alice to observe a, given that Bob has observed b, is

$$p(a|b) = \frac{p(a, b)}{\sum_{a=1}^{d_A} p(a, b)}$$

We say that the observations of Alice and Bob are uncorrelated if (and only if)

$$p(a|b) = p(a|b') \qquad \forall a \in \{1, ..., d_A\} \ , \ b, b' \in \{1, ..., d_B\} \tag{1}$$

In other words, the probability for Alice to observe a does not depend on whether Bob observes b or b' (Alice's observations are *independent* of Bob's).

Part I (10 points): show that the expression of conditional independence in (1) is symmetric between Alice and Bob (if the probability for Alice's observations do not depend on Bob's observations, then the probability of Bob's observations do not depend on Alice's observations). Show that either of these conditions implies

$$p(a, b) = p(a)p(b) \quad , \quad \forall a \in \{1, ..., d_A\}, b \in \{1, ..., d_B\}. \tag{2}$$

Therefore we may freely take either (1) or (2) as the definition of uncorrelated events (and if either of these conditions is violated then the events are correlated, by definition).

Part II (10 points): We can represent the joint distribution of Alice and Bob as a probability distribution over quantum states i.e. a density matrix:

$$\rho_{AB} = \sum_{a,b=1}^{d_A,d_B} p(a, b)|ab\rangle\langle ab| = \sum_{a,b=1}^{d_A,d_B} p(a, b) \left(|a\rangle\langle a| \otimes |b\rangle\langle b|\right), \tag{3}$$

where the last form emphasizes the fact that $\{|a\rangle\langle a| \otimes |b\rangle\langle b|\}_{a,b=1}^{d_A,d_B}$ is a *tensor product basis* for $\mathcal{H}_A \otimes \mathcal{H}_B$. Using the results from Part I, show that if the observations made by Alice and Bob are uncorrelated then $\rho = \rho_A \otimes \rho_B$. Can the state ρ_{AB} in (3) be entangled (across the bipartition A, B) if the observations of Alice and Bob are correlated?

Part III (30 points): Prove or disprove. A pure state $|\psi_{AB}\rangle$ is entangled (across the bipartition A, B) if and only if for every *tensor product basis* on A, B the measurement outcomes on subsystem A are correlated with the measurement outcomes on subsystem B.

Problem 2: A distance measure between quantum states. If $p, \tilde{p}$ are probability distributions on the event space $\Omega = \{1, ..., N\}$ then the total variation distance between p and q is

$$d(p, \tilde{p}) = \frac{1}{2} \sum_{x=1}^{N} |p_x - \tilde{p}_x|. \tag{4}$$

Part I (10 points): Prove that definition (4) is equivalent to

$$d(p, \tilde{p}) = \max_{A \subseteq \Omega} |p(A) - \tilde{p}(A)| \tag{5}$$

where $p(A) = \sum_{x \in A} p_x$ (hint: consider the subset of events $A = \{x : p_x \geq \tilde{p}_x\}$). From (5) we have the operational interpretation that the total variation distance quantifies the maximum difference in the probability assigned to any two (collections of) events.

Part II (20 points): Suppose that distributions p, $\tilde{p}$ arose by measuring two density matrices $\rho, \tilde{\rho}$ in some complete basis of states $\{|\psi_a\rangle\}_{a=1}^{N}$, where the measurement Kraus operators $E_a = |\psi_a\rangle\langle\psi_a|$ satisfy the completeness relation

$$\sum_{a=1}^{N} E_a^\dagger E_a = I.$$

Show that

$$d(p, \tilde{p}) \leq \frac{1}{2} \sum_{i=1}^{N} |\lambda_i| \tag{6}$$

where $\lambda_i, i = 1, ..., N$ are the eigenvalues of the operator $\rho - \tilde{\rho}$. Find a choice of measurement operators $\{E_a\}$ that saturates the upper bound in (6).

Part III (20 points): The upper bound (6) supplies an operational meaning to the use of

$$d(\rho, \tilde{\rho}) = \sum_{i=1}^{N} |\lambda_i|$$

as a distance measure between quantum states, because it is an upper bound on the maximum difference in probability that $\rho, \tilde{\rho}$ can assign to any two events, in any choice of measurement basis. Show that this distance measure can be expressed in terms of the *trace norm*,

$$\|A\|_{\mathrm{Tr}} = \mathrm{tr}\left(\sqrt{A^\dagger A}\right)$$

for some operator A, which leads this measure to be called the "trace distance." Combine the results of part I and II to show that

$$d(\rho, \tilde{\rho}) = \max_{P} \mathrm{tr}\left(P\left(\rho - \tilde{\rho}\right)\right)$$

where the maximum is taken over all projectors P.

Problem 3: Quantum channels. Recall that every quantum channel $\mathcal{E}$ that maps valid density matrices to valid density matrices has a Kraus operator-sum representation:

$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad , \quad \sum_k E_k^\dagger E_k = I.$$

Part I (10 points): Let $|\psi\rangle$ be an n-qubit state. Find Kraus operators that map any n-qubit quantum state to $|\psi\rangle\langle\psi|$. This quantum channel could model a relaxation or decay process to the state $|\psi\rangle$.

Part II (10 points): Find Kraus operators that map any n-qubit quantum state to the maximally mixed state $\rho = 2^{-n} I$.

Part III (10 points): Show that the composition of quantum channels is a quantum channel. If

$$\mathcal{E}(\rho) = \sum_{k=1}^{S} E_k \rho E_k^\dagger \quad , \quad \mathcal{E}'(\rho) = \sum_{k=1}^{T} E'_k \rho E_k'^\dagger,$$

then what are the Kraus operators for $\mathcal{E}' \circ \mathcal{E}$?

Part IV (10 points): For a composite system $|\psi_{AB}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, find the Kraus operator-sum representation of the partial trace:

$$\mathcal{E}(\rho_{AB}) = \mathrm{tr}_B \rho_{AB}$$

In this example, pay particular attention to the fact that the Kraus operators map states in a higher dimensional Hilbert space to states in a lower dimensional Hilbert space.

Part V (10 points): Going in the opposite direction, construct the Kraus operators for a channel that maps any state ρ_A on the Hilbert space $\mathcal{H}_A$ to the state $\rho_A \otimes \rho_B$ for some fixed state ρ_B on the Hilbert space $\mathcal{H}_B$.

Part VI (20 points): Consider the bit flip channel, which acts on a single qubit and does nothing with probability $1 - p$, and interchanges $|0\rangle$ and $|1\rangle$ with probability p.

(a) Find the Kraus operator-sum representation of this channel.

(b) If the system is initially in the state $|\psi\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$, then what is the state of the system after one application of the bit flip channel?

(c) Find a unitary channel acting on the system and an environment such that tracing out the environment yields the bit flip channel on the system. Justify this form by deriving the correct Kraus operators E_k by acting with a joint unitary and applying the partial trace.

Problem 4: Noisy classical channel. Let X be a binary message source (a random variable distributed over $\{0,1\}$). We can model a noisy communication channel $\mathcal{N}$ with input X by another random variable Y with a distribution of outputs $y \in \{0,1\}$ conditioned on inputs x,

$$\mathcal{N}: \quad p(y|x).$$

In class we saw a proof sketch of Shannon's noiseless channel coding theorem, which says that the optimal rate of message compression for communication over a noiseless channel is given in terms of the entropy of the source. Shannon's second breakthrough theorem states that the capacity of a noisy channel (the highest rate with which it can transmit information) is given in terms of the mutual information between the input and the output of the channel,

$$C = I(X:Y)$$

We will postpone the derivation of this result until we are ready to discuss its quantum counterpart later in the course, but for now we can gain some understanding of this result from an example.

Consider a *binary symmetric channel*, which takes as input messages x from the set $\{0,1\}$ with probabilities $\{0.5, 0.5\}$, and outputs messages y with the following conditional probabilities:

$$p(y=1|x=1) = 1-\epsilon \ \ , \ \ p(y=0|x=1) = \epsilon \ \ , \ \ p(y=1|x=0) = \epsilon \ \ , \ \ p(y=0|x=0) = 1-\epsilon$$

for some $0 \leq \epsilon \leq 1$. This channel represents faithful transmission with probability $1-\epsilon$, and a bit flip with probability ϵ.

Part I (5 points): What is the entropy of the source, $S(X)$?

Part II (5 points): What is the probability distribution of the outputs, $p(y)$, and what is the entropy of this distribution, $S(Y)$?

Part III (5 points): What is the joint distribution $p(x,y)$ over inputs and outputs, and what is the joint entropy $S(X,Y)$?

Part IV (5 points): What is the mutual information $I(X:Y)$?

Part V (5 points): For what value(s) of ϵ is the mutual information, and hence the channel capacity, maximal? Does this make sense?

Part VI (5 points): For what value(s) of ϵ is the mutual information, and hence the channel capacity, minimal? Does this make sense?