Physics 572: Homework #4

Due Date: April 30$^{\text{th}}$, 2019

Problem 1: Marginal Consistency. The NP-hardness of the classical marginal consistency problem can be seen from a reduction to the problem 3-COLORING. Let $G = (V, E)$ be a combinatorial graph with vertex set $V = \{1, ..., |V|\}$. For each vertex $u \in V$ we consider a random variable X_u taking values in the set $\{r, g, b\}$. For each edge $(u, v) \in E$, the joint distribution $p_{uv}(X_u, X_v)$ is taken to be uniform over the set $\{r, g, b\}^2 - \{rr, gg, bb\}$. Let $P = \{p_{uv} : (u, v) \in E\}$ denote the set of these joint distributions defined on edges. We say that P is a consistent set of marginals iff there exists a joint distribution $p(X_1, ..., X_{|V|})$ with

$$p_{uv}(X_u, X_v) = \sum_{\{X_q\}_{q=1}^{|V|} - \{X_u, X_v\}} p(X_1, ..., X_{|V|})$$

I. (20 points) Show that P is a consistent set of marginals if and only if G is 3-colorable.

II. (20 points) Suppose Merlin wants to convince Arthur that the graph is 3-colorable, without giving him any clue about how to color it. This is possible if we can force Merlin to "commit" to certain choices. We play a game with multiple rounds. In each round Merlin commits to a labeling, and also performs a random permutation of the colors $\{r, g, b\}$. Arthur chooses a random edge and challenges Merlin to reveal the colors of its vertices. Merlin is forced to reveal the colors he committed to. On each round Merlin permutes the colors randomly (and chooses a different labeling, if he wants), and then commits again. This is called a "zero knowledge protocol" for graph coloring, and the commitment can be done by "digital signing", or you can think of it as enforced by a referee. If the game goes for L rounds, what is the soundness of this ZK protocol (the maximum probability to falsely conclude the graph is 3-colorable, when it really isn't) ?

III. (20 points) Consider qubits A, B, C with density matrices ρ_{AB}, ρ_{BC}. A state ρ_{ABC} is a symmetric extension of ρ_{AB}, ρ_{BC} if $\rho_{AB} = \text{tr}_C \rho_{ABC}, \rho_{BC} = \text{tr}_A \rho_{ABC}$, and $\rho_{AB} = \rho_{AC}$. A necessary and sufficient condition for the existence of such a symmetric extension is

$$\text{tr}(\rho_B^2) \geq \text{tr}(\rho_{AB}^2) - 4\sqrt{\det(\rho_{AB})}. \tag{1}$$

Find the value of $p \geq 0$ that makes (1) an equality for the state ρ_{AB} given by

$$\rho_{AB} = \frac{(1-p)}{4} I + p|\Phi^+\rangle\langle\Phi^+| \quad , \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2}$$

IV. (20 points) Let ρ_{AB} be a full rank state that saturates (1) and define

$$H_{AB} = \sqrt{\det(\rho_{AB})}\rho_{AB}^{-1} + \rho_{AB} - \rho_B. \tag{3}$$

Show that $\text{tr}(H_{AB}\rho_{AB}) = 0$.

V. (20 points) Define H_{AC} by analogy with (3), and show that the maximally mixed state in the ground space of ρ_{ABC} of $H = H_{AB} + H_{AC}$ for the example computed in part III is a symmetric extension of ρ_{AB} in equation (1).

Problem 2: History states and tensor networks. Consider a quantum circuit described by a sequence of gates $U_1, ..., U_T$ acting on an input state $|0^n\rangle$. The Feynman-Kitaev history state for this circuit is a superposition over time steps of this circuit, which are entangled together with a "clock register" (a qudit) denoted by $|t\rangle$, with $t = 0, ..., T$,

$$|\Psi_{\text{history}}\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} U_t \dots U_1 |0^n\rangle \otimes |t\rangle \tag{4}$$

Define $H = H_{\text{in}} + H_{\text{prop}}$, with $H_{\text{in}} = (\sum_{i=1}^{n} |1\rangle\langle 1|_i) \otimes |0\rangle\langle 0|$ and $H_{\text{prop}} = \sum_{t=1}^{T} H_{\text{prop}}(t)$, where

$$H_{\text{prop}}(t) = \frac{1}{2}\Big(I \otimes |t\rangle\langle t| + I \otimes |t-1\rangle\langle t-1| - U_t \otimes |t\rangle\langle t-1| - U_t^\dagger \otimes |t-1\rangle\langle t|\Big), \tag{5}$$

so that $H|\Psi_{\text{history}}\rangle = 0$ and $|\Psi_{\text{history}}\rangle$ is the unique ground state of H.

I. (10 points) Show that the following operator W is unitary:

$$W = \sum_{t=0}^{T} U_t \dots U_1 \otimes |t\rangle\langle t| \tag{6}$$

II. (30 points) Prove that $W H_{\text{prop}} W^\dagger = I \otimes H_{\text{path}}$, where

$$H_{\text{path}} = \frac{1}{2} \sum_{t=1}^{T} (|t\rangle\langle t| + |t-1\rangle\langle t-1| - |t\rangle\langle t-1| - |t-1\rangle\langle t|) \tag{7}$$

III. (30 points) Diagonalize the Hamiltonian H_{path} by using a change of basis (which can be seen as a quantum Fourier transform, or a change of basis from position to momentum),

$$|t\rangle \to \frac{1}{\sqrt{T+1}} \sum_{k=-T/2}^{k=+T/2} \exp\left(\frac{2\pi i k t}{T+1}\right) |k\rangle \tag{8}$$

IV. (30 points) To embed the history state into a qubit system, the standard trick is to express the states of the clock qudit in unary. In this representation, the uniform superposition of clock states (which is the ground state of H_{path}) takes the form

$$|s\rangle = \frac{1}{\sqrt{T+1}} \sum_{t=0}^{T} |1^t 0^{T-t}\rangle$$

Show that this superposition state $|s\rangle$ can be represented as a matrix product state,

$$|s\rangle = \sum_{(s_1,...,s_n)\in\{0,1\}^{T+1}} A^{(s_0)} A^{(s_1)} ... A^{(s_T)} |s_0, ..., s_T\rangle$$

where the matrices $A^{(s_i)}$ have a constant bond dimension (you may handle the boundary conditions as you prefer, one choice is to make $A^{(s_0)}, A^{(s_T)}$ rectangular matrices).