Quantifying Entanglement

For a general quantum state ρ , the Schumacher compression rate provides an operational meaning to $S(\rho)$.

Recall that the von Neumann entropy also measures entanglement in a bipartite state $|\psi_{AB}\rangle$, since

$$I(A:B) = S_A + S_B - S_{AB} = 2S_A = 2S_B$$

What is the operational meaning of entanglement entropy? Why is it a good measure of entanglement?

Recall that our first measure of entanglement was the Schmidt rank, but the flaw in this measure is that it is not robust. For example the unentangled state $|\psi_0\rangle = |00\rangle$ has Schmidt rank 1, while

$$|\psi_\epsilon\rangle = \sqrt{1-\epsilon^2}|00\rangle + \epsilon|11\rangle$$

Has Schmidt rank 2 for every $\epsilon > 0$, but the fidelity between these states is $|\langle\psi_0|\psi_\epsilon\rangle|^2 = 1 - \epsilon^2$. Therefore the two states can be arbitrarily close together while one is entangled and the other is not. This means the Schmidt rank is a very rough (discontinuous) measure of entanglement.

Quantifying Entanglement

If Alice and Bob are restricted to local operations and classical communication, then entanglement is a limited resource (because they are restricted to applying separable quantum channels).

From this perspective, we want measures of the entanglement shared in Alice and Bob's initial state to tell us about protocols that they could perform using LOCC.

This again indicates a shortcoming of the Schmidt rank. The state $|\psi_\epsilon\rangle = \sqrt{1-\epsilon^2}|00\rangle + \epsilon|11\rangle$ has the same Schmidt rank as a Bell pair, but will not be a useful resource for teleportation as $\epsilon \to 0$.

Suppose Alice and Bob share many initial Bell pairs. Then they may be interested in the amount of Bell pairs needed to prepare a particular entangled state $|\psi_{AB}\rangle$ using only LOCC, which is called the **entanglement cost**.

Conversely, if Alice and Bob initially share $|\psi_{AB}\rangle$, then they may wish to convert the entanglement in $|\psi_{AB}\rangle$ to standardized entangled states like Bell pairs, and the amount of ebits they obtain is the **distillable entanglement**.

Quantifying Entanglement

To relate the entanglement cost and the distillable entanglement to the entanglement entropy, we will need to work in the asymptotic limit in which we have many identical copies of the state, $|\psi_{AB}\rangle^{\otimes n}$.

A rate *R* of (LOCC) conversion from Bell pairs $|\Phi^+\rangle$ to the state $|\phi\rangle$ is asymptotically achievable if for any $\epsilon, \delta > 0$ and all sufficiently large *n* there is a *k* satisfying

$$\frac{k}{n} \leq R + \delta$$

Such that $|\Phi^+\rangle^{\otimes k}$ can be mapped by LOCC to a state which has fidelity $1 - \epsilon$ with the target state $|\phi\rangle^{\otimes n}$.

With this we can rigorously define entanglement cost:

$$E_C(|\psi\rangle) = \inf\{\text{achievable rate for creating } |\psi\rangle \text{ out of Bell pairs}\}$$

Asymptotically, we create many copies of $|\psi\rangle$ by using *R* Bell pairs per copy.

Quantifying Entanglement

The distillable entanglement is similarly defined in terms of an asymptotically achievable conversion rate.

$|\psi_{AB}\rangle^{\otimes n}$

A rate *R* of (LOCC) conversion from the state $|\phi\rangle$ to Bell pairs $|\Phi^+\rangle$ is asymptotically achievable if for any $\epsilon, \delta > 0$ and all sufficiently large *n* there is a *k* satisfying

$$\frac{k}{n} \geq R - \delta$$

Such that $|\phi\rangle^{\otimes n}$ can be mapped by LOCC to a state which has fidelity $1 - \epsilon$ with the target state $|\Phi^+\rangle^{\otimes k}$.

And from this we have the rigorous definition of distillable entanglement:

$$E_D(|\psi\rangle) = \inf\{\text{achievable rate for creating Bell pairs out of } |\psi\rangle\}$$

Quantifying Entanglement

Since LOCC can generate entanglement, our first observation is that for any state $|\psi\rangle$ we have

$$E_D(|\psi\rangle) \leq E_C(|\psi\rangle)$$

Otherwise Alice and Bob would be able to generate an unlimited number of Bell pairs using LOCC by converting back and forth between copies of $|\psi\rangle$ and Bell pairs.

In fact, the protocol we will describe that relates these entanglement conversion rates to the von Neumann entropy is asymptotically reversible, and so we will find

$$E_D(|\psi\rangle) = E_C(|\psi\rangle) \quad \forall |\psi\rangle$$

However in the mixed state case this is no longer true, and the distillable entanglement may be strictly less than the entanglement cost. But for pure states we can drop the subscript on $E(|\psi\rangle)$, and we will show

$$E(|\psi_{AB}\rangle) = S(\rho_A)$$

Quantifying Entanglement

Just as Shannon entropy allows us to measure fractions of a bit, we will find it convenient in the asymptotic setting to let the entanglement entropy measure fractions of a ebit.

If Alice and Bob each hold a quantum system of Hilbert space dimension D, then the maximally entangled state is:

$$\frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$$

Which has entanglement entropy log d. We'll refer to this as "log d Bell pairs" even when d is not a power of 2.

Therefore our proof strategy to show $E_D(|\psi_{AB}\rangle) = E_C(|\psi_{AB}\rangle) = S(\rho_A)$ for all bipartite pure states is as follows.

Alice and Bob will use a maximally entangled state with dimension $d = 2^{n(S_A+\delta)}$ together with LOCC to create n copies of $|\psi\rangle$, with arbitrarily good fidelity as $n \to \infty$.

Conversely, Alice and Bob will use n copies of $|\psi\rangle$ together with LOCC to create a maximally entangled state with dimension $d = 2^{n(S_A-\delta)}$, with arbitrarily good fidelity as $n \to \infty$.

Quantifying Entanglement

First, Alice and Bob will use a maximally entangled state with dimension $d = 2^{n(S_A+\delta)}$ together with LOCC to create n copies of $|\psi\rangle$, with arbitrarily good fidelity as $n \to \infty$.

They will accomplish this by combining Schumacher compression with quantum teleportation.

Alice will begin by making several copies of a state $|\psi_{AC}\rangle$ in her lab, where C is the subsystem that will be compressed and then teleported to Bob with LOCC to form the state $|\psi_{AB}\rangle$.

If A and C are both d-dimensional quantum states, then the Schmidt decomposition is

$$|\psi_{AC}\rangle = \sqrt{p(0)}|00\rangle + \sqrt{p(1)}|11\rangle + ... + \sqrt{p(d-1)}|d-1, d-1\rangle$$

Therefore in the asymptotic limit of Alice holding many identical copies of the state,

$$|\psi_{AC}\rangle^{\otimes n} = \sum_{x_1,\dots,x_n=0}^{d-1} \sqrt{p(x_1)...p(x_n)}|x_1...x_n\rangle_{A^n}|x_1...x_n\rangle_{C^n} = \sum_{\vec{x}} \sqrt{p(\vec{x})}|\vec{x}\rangle_{A^n}|\vec{x}\rangle_{C^n}$$

Quantifying Entanglement

In her laboratory, Alice will project this state into a δ- typical subspace, and this succeeds with high probability:

$$P = \sum_{\delta\text{-typical } \vec{x}} p(\vec{x}) \geq 1 - \epsilon$$

The state after projecting onto the typical subspace is

$$|\tilde{\psi}\rangle_{A^n C^n} = P^{-1/2} \sum_{\delta\text{-typical } \vec{x}} \sqrt{p(\vec{x})} |\vec{x}\rangle_{A^n} |\vec{x}\rangle_{C^n}$$

Which has a high overlap with the original state since

$$\left(\langle \psi_{AC}|^{\otimes n}\right) |\tilde{\psi}\rangle_{A^n B^n} = P^{-1/2} \sum_{\delta\text{-typical } \vec{x}} p(\vec{x}) \geq \sqrt{1 - \epsilon}$$

Quantifying Entanglement

Since the typical subspace has dimension at most $2^{n(S_A+\delta)}$, Alice can teleport her C^n part of the state to Bob using at most $n(S_A + \delta)$ Bell pairs that were initially shared between Alice and Bob.

The generalization of teleportation to d-dimensional systems where d is not a power of 2 was already addressed in the original paper by Bennet et al. One replaces the Bell state with a maximally entangled state, and then performs a measure over all maximally entangled bases. So as before this only requires LOCC.

After Bob receives the classical information from Alice needed to correct the teleported state, he can decompress it so that the state they share has high fidelity with $|\psi_{AB}\rangle^{\otimes n}$, as intended.

This protocol demonstrates that the entanglement cost $E(|\psi_{AB}\rangle)$ is no larger than S_A .

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This protocol demonstrates that the entanglement cost $E_C(|\psi_{AB}\rangle)$ is no larger than S_A. Therefore if we show that the distillable entanglement satisfies $E_D(|\psi_{AB}\rangle) \geq S_A$, then by $E_D(|\psi\rangle) \leq E_C(|\psi\rangle)$ we'll show

$$E_D(|\psi_{AB}\rangle = E_C(|\psi_{AB}\rangle) \equiv E(|\psi_{AB}\rangle) = S_A$$

Quantifying Entanglement

For entanglement distillation, a single copy of $|\psi_{AB}\rangle$ may only be partially entangled, like the example

$$|\psi(p)\rangle = \sqrt{1-p}|00\rangle + \sqrt{p}|11\rangle$$

Therefore given many copies of this state $|\psi_{AB}\rangle^{\otimes n}$, the entanglement may be *diluted* throughout the system, and the goal for Alice and Bob is to *concentrate* this entanglement in order to *distill* Bell pairs.

The goal will be to show that they can squeeze all of this entanglement into $n(S_A - \delta)$ Bell pairs.

To illustrate entanglement distillation in general, consider many copies of the example state $|\psi(p)\rangle$ (which is already in the form of a Schmidt decomposition)

$$|\psi(p)\rangle^{\otimes n} = \left(\sqrt{1-p}|00\rangle + \sqrt{p}|11\rangle\right)^{\otimes n} = \sum_{x\in\{0,1\}^n} \sqrt{P(x)}|x\rangle|x\rangle$$

Where $P(x)$ only depends on the Hamming weight (the number of 1's) in the string x.

Quantifying Entanglement

Suppose Alice uses a quantum channel to measure the Hamming weight of her part of the state, without revealing any other information about the state:

$$|\psi(p)\rangle^{\otimes n} = \sum_{x\in\{0,1\}^n} \sqrt{P(x)}|x\rangle|x\rangle$$

There are $\binom{n}{m}$ terms in Alice's state which have Hamming weight *m*, each with probability $(1-p)^{n-m}p^m$, so

$$P(m) = \binom{n}{m}(1-p)^{n-m}p^m$$

After the measurement, Alice will be left with the uniform superposition of strings with Hamming weight m. Since Bob's bit strings are perfectly correlated with Alice's, the state they now hold is

$$\sum_{x\in\{0,1\}^n:|x|=m} |x\rangle_A|x\rangle_B$$

Which is a maximally entangled state with dimension $\binom{n}{m}$!

Quantifying Entanglement

In the asymptotic limit, the Hamming weight will be near m = n p with high probability.

Therefore we can apply Stirling's approximation to the dimension of this maximally entangled state, $d = \binom{n}{np}$, to recover a result that is by now familiar:

$$2^{n(S(p)-o(1))} \leq d \leq 2^{n(S(p)+o(1))}$$

With high probability, where $S(p) = -(1-p)\log(1-p) - p\log p$ is the binary entropy function.

This proves that $E_D(|\psi(p)\rangle) \geq S(p)$. The generalization to d-dimensional systems is relatively straightforward: now instead of measuring the number of 0's and 1's, Alice measures the number of 0's, 1's, 2's,… etc. This projects onto the permutation symmetric subspace of strings with that number 0's, 1's, 2's,… etc.

The dimension of this subspace is counted by a multinomial coefficient, and as before in the asymptotic limit this dimension is counted by entropy. So it is exactly analogous to going from a binary alphabet to a general finite alphabet in our discussion of Shannon theory.

Quantifying Entanglement

In the case of mixed states ρ_{AB}, we can have a strict separation $E_D(\rho) < E_C(\rho)$.

In fact this should be expected, because Bell pairs are pure and the state is mixed. As we go from a pure state to a mixed state, some information must be lost, and this is in general irreversible.

One can even have mixed states with a nonzero entanglement cost, and zero distillable entanglement. This phenomenon is called *bound entanglement*.

The separation between entanglement cost and distillable entanglement also means the entanglement entropy no longer neatly characterizes them both, and this remains an active subject of research.

In response the community has introduced a variety of entanglement measures called entanglement monotones.

In general an entanglement monotone should be zero for separable states, invariant under local reversible operations, and nonincreasing under general LOCC channels.

Quantifying Entanglement

Another striking difference between quantum and classical correlations is that quantum correlations carry a notion of exclusivity: **entanglement is monogamous**.

Suppose Alice shares a Bell pair with Bob. Can the qubits in this Bell pair be entangled with a third qubit C?

No, because by assumption the reduced state ρ_{AB} is pure. This tradeoff can also be made quantitative, so that any entanglement shared between A and B restricts the amount of entanglement that can also be shared with C.

This contrasts with classical correlation, that can be shared amongst multiple parties with no restrictions.

Quantifying Entanglement

Blackhole information paradox: attempts to combine general relativity and quantum mechanics have led to puzzles concerning the fate of quantum information that falls into a black hole.

In 1975, Hawking and Bekenstein showed that quantum mechanical effects cause black holes to radiate energy, but their calculations suggested this energy had no correlation with the original in-falling matter.

Fundamentally irreversible information loss would be a problem for our belief that time evolution is unitarity, though we might hope for this to be solved by a quantum theory of gravity.

In 2012, a new sharper version of the black hole information paradox called the “firewall paradox” was proposed, and it is based on the monogamy of entanglement.

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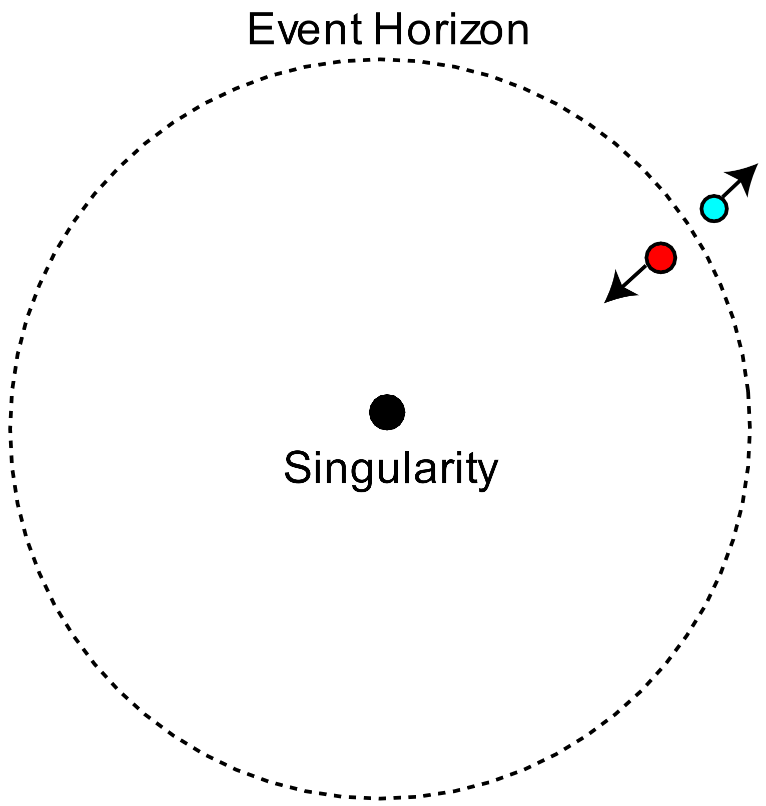

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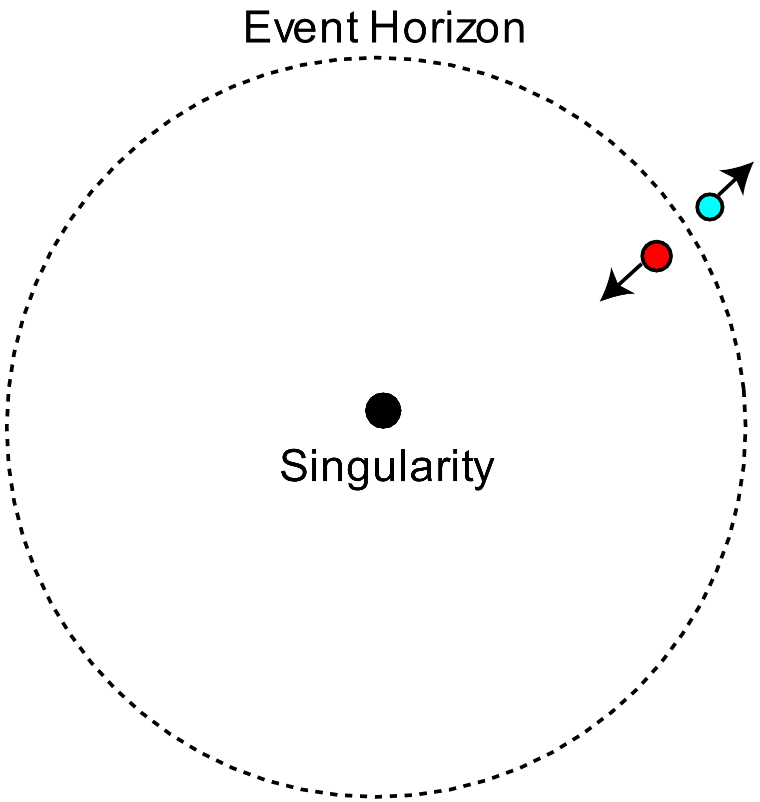
Quantifying Entanglement

In 2012, a new sharper version of the black hole information paradox called the “firewall paradox” was proposed, and it is based on the monogamy of entanglement.

The idea is that outgoing Hawking quanta are entangled with infalling Hawking quanta. But after more than half of the mass of the black hole has radiated away, it should also be the case that the outgoing particle is entangled with Hawking radiation from the distant past. This contradicts monogamy of entanglement.



The term “firewall” refers to a possible resolution, where one imagines a fiery wall of destruction at the event horizon which breaks the entanglement between the Hawking pairs. But this conflicts with our understanding that the event horizon should be smooth and unremarkable. So it proposes a modification of quantum field theory.