

Sudarshan's non-relativistic approach to the spin-statistics connection

Anil Shaji

Information Physics Group, Department of Physics and Astronomy, University of New Mexico, Albuquerque NM

E-mail: shaji@unm.edu

Abstract. George Sudarshan was quite a bit ahead of the times when he sketched out a non-relativistic proof of the spin-statistics connection forty years back. A recent revival of interest in the role of relativistic arguments in proving the spin-statistics connection prompted us to revisit Sudarshan's original arguments and present them in simpler fashion. The proof is designed to work in three space dimensions. Sudarshan's observation was that if we impose $SU(2)$ invariance, along with a few other restrictions, on the kinematic part of the Lagrangian of a field theory which is derivable from a Weiss-Schwinger type of the Principle of Least Action, then the spin-statistics connection automatically follows. The assumptions and simplifications that go into constructing the proof will be carefully laid out so as to identify the limitations of this approach.

1. Introduction

Is it possible to understand the spin-statistics connection without using relativistic arguments? A few years back this might have been considered merely a pedagogical question; perhaps even a misguided one. After all, Pauli, in his seminal paper on this topic [1], considers his proof of the spin-statistics theorem a triumph of the special theory of relativity and says:

In conclusion we wish to state, that according to our opinion the connection between spin and statistics is one of the most important applications of the special relativity theory.

Despite Pauli's assertions and inspired by Prof. Sudarshan's ideas [2, 3, 4, 5], we venture to ask what really is the role of special relativity (if any) in understanding the connection between spin and statistics? Is it possible to deduce the theorem starting from just the postulates of quantum theory of fields without involving special relativity?

In Pauli's original proof and subsequent elaborations and completions of it [6, 7, 8, 9], relativity enters in the following three ways.

- (i) Quantum fields are considered to be finite dimensional representations of the Lorentz group.
- (ii) Observables associated with these fields vanish at space like separations.
- (iii) The quantized fields have positive energy and charge density.

Unfortunately none of these three assertions have a direct counterpart in non relativistic quantum field theories. Even energies need not be positive, they just have to be bounded

from below. What we recognize here is that somehow adapting Pauli's approach for showing the spin-statistics connection and then taking the non-relativistic limit of such a formulation is very unlikely to give us anything meaningful. This also means that if one were to follow such an approach, it will be easy to conclude rather prematurely that non-relativistic proofs of the spin-statistics connection cannot exist [10]. The question we have to ask then is whether there is an alternate starting point from which we can deduce the spin statistics connection. A starting point which has very little, and if possible, nothing to do with relativistic considerations.

What is our motivation for trying to prove the spin-statistics connection without using special relativity? It comes from experimentally accessible, non-relativistic, few-body, systems that require the spin-statistics relation to explain their behavior (see for instance [11] and references therein). We note here that the spin-statistics connection (equivalently the Pauli exclusion principle) was originally postulated in the context of atomic and nuclear physics. The original proof by Pauli and others were also developed with this domain of applicability in mind. It turned out that the domain of applicability of the theorem is much larger than that, extending in both directions to particle physics as well as to semi-classical quantum systems.

The spin-statistics connection seems crucial to understanding the behavior of several physical systems for which relativistic considerations seem quite insignificant. In fact, relativistic considerations are not just insignificant, but sometimes including them makes the understanding quite complicated. Examples of such systems include electrons in metals, phonons in solids and Bose condensates, atoms in optical lattices, Cooper pairs, entangled states and so on. Simply put, it seems that understanding the spin-statistics connection without relativity can give us better insights into the nature of such systems than going the other way and constructing a fully relativistic theory of these systems to include the spin-statistics connection in a natural way.

Spectacular advances in experimental techniques over the past two decades or so, driven partly by the advances in quantum information theory [12], means that quantum systems like the ones above are not just things that we observe in nature but rather systems that can be constructed, controlled and manipulated at will. Non-relativistic theories seem to adequately describe most of these systems and spin-statistics connection has to be inserted "by hand" when formulating these theories. With increasing experimental control over these systems, it is now reasonable to ask if we can at all tailor make a quantum system in which quasi-particles exhibit the wrong spin-statistics connection. This is not as unreasonable as it sounds because, after all, anyonic quasi-particles is quite actively talked about in the context of topological quantum computation [13, 14]. It is worth mentioning that in quantum information theory, the spin-statistics connection is often treated as an aid in doing certain information processing jobs [15].

Several approaches have been tried to obtain the spin-statistics connection without using relativity [16, 17, 18, 19]. We do not mention them here; instead, refer the reader to [4, 5] for detailed discussions on many of them.

This paper is structured as follows: In Sec. 2 we outline the steps needed to prove the spin-statistics connection using Sudarshan's approach. Sections 3, 4, 5, and 6 contain the details of the proof. We discuss briefly the implications of the assumptions that led to the proof in Sec. 7 and concluding remarks are in Sec. 8.

2. Outline of the proof

Our proof of the spin-statistics connection uses second quantization and field theory for describing quantum many body systems. We have to be careful though to make sure that the fields we consider could very well be non-relativistic ones. By using field operators in our discussion we can describe particle exchange that is central to the spin-statistics relation in very general terms. We carefully and deliberately avoid the elaborate devices based on rotation operators and adiabatic transport used by some authors to obtain the spin-statistics connection. This is because we believe that the spin-statistics connection must arise from the properties of

the physical system independent of the *processes* that it goes through.

The objective is to establish a relationship between two distinct properties associated with quantum fields denoted by ξ :

- **Spin:** The spin is specified by how the field transforms under space rotations.
- **Statistics:** The statistics depends on whether the fields obey commutation or anti-commutation relations.

Starting from the spin we relate it to the statistics that the quantum field should obey through the steps that are outlined below.

- (i) Continuing the theme of behavior under rotations, we construct a Lagrangian density \mathcal{L} for the fields that is a scalar with a few other restrictions on its kinematic part.
- (ii) Once we have the Lagrangian, we make an action integral I from it and then using a Weiss-Schwinger [20, 21] type of least action principle we construct a complete theory of the non-relativistic fields we are interested in.
- (iii) In this generalized principle of least action the surface variation of the action is the generator of transformations in the field quantities themselves [20]. This lets us write down the following equation involving products of fields, their variations, and other quantities that appear in \mathcal{L}

$$[\xi^{(n)}, \delta I_{\delta\xi}] = \delta\xi^{(n)}. \quad (1)$$

- (iv) We simplify the above equation assuming either commutation or anti-commutation relations for the fields ξ . We then find that if the initial assertion that the Lagrangian be a scalar is to hold then the commutation and anti-commutation relations have to go along with one type of spin or the other.

In other words, consistency checks on this sequence of steps gives us the spin-statistics relation. In the next three sections we flush out the details of these steps and make sure that the we have not used any relativistic considerations in each step.

3. The Lagrangian

In what follows it is assumed that we are dealing with a standard version of quantum field theory which is derivable from a Principle of Least Action based on a Lagrangian. Our arguments leading to the spin-statistics connection are confined at the outset to *three space dimensions*. The three dimensional space admits multicomponent wave functions that have specific transformation properties under rotations. The rotations that we consider are not restricted to the group of proper rotations in three dimensional space, $SO(3)$, but rather to its covering group $SU(2)$. The extension to $SU(2)$ is needed for our discussion of the spin-statistics connection because we have to include both integral and half-integral spin representations of the rotation group and be able to treat them as proper representations. Since $SU(2)$ is a subgroup of the covering group of the Galilei group, we are able to keep our arguments entirely non-relativistic.

Along the lines of Schwinger's proof [20] of the spin-statistics connection the following four conditions are imposed on the *kinematic part* of the Lagrangian density.

- (i) The Lagrangian \mathcal{L} is *invariant under $SU(2)$ (and $SO(3)$) transformations* and corresponds to a field theory for fields, ξ , which are finite dimensional representations of the $SU(2)$ covering of the three dimensional rotation group.
- (ii) The Lagrangian is expressed in the *hermitian field basis*; $\xi = \xi^\dagger$.
- (iii) It is at most linear in the *first time derivatives* of the field and the time derivatives occur only in the kinematic term.
- (iv) The *kinematic term is bilinear* in the field ξ

The last two requirements ensure that the Euler-Lagrange equations of motion derived from the Lagrangian are first order linear differential equations of the Hamiltonian form. The Dirac Lagrangian is already in the required form while for massive spin-0 and spin-1 fields satisfying the Klein-Gordon equation, the Duffin-Kemmer [22, 23] form of the equation is such a suitable one. A short discussion on how to convert a generic bilinear kinematic Lagrangian with higher order time derivatives in it into a form that is suitable for the present discussion is given in [24].

We now focus on the kinematic part of the Lagrangian written in a form that satisfies the four requirements laid out above. This is acceptable because we do not expect the spin-statistics connection to come out of interactions and other dynamical effects. The generic (Schwinger) Lagrangian satisfying the four requirements has the form

$$\begin{aligned} \mathcal{L} = & \frac{1}{2} K_{rs}^0 (\xi^{(r)} \dot{\xi}^{(s)} - \dot{\xi}^{(r)} \xi^{(s)}) \\ & + \text{Terms containing space derivatives} + \text{Mass terms} \\ & + \text{Other non-derivative terms including interactions.} \end{aligned} \quad (2)$$

The first term that is explicitly written down in the *kinematic* term. In the Lagrangian, r and s are spin indices that are summed over. The kinematic term is explicitly antisymmetrized with respect to the time derivative so that we avoid the possibility of the two terms together forming a total derivative. The terms containing the space derivatives need not be of this particular form and can contain higher derivatives and higher powers of the derivatives. No restrictions are placed on the mass term or on any interaction terms that may be present except to the extent that condition (iii) rules out velocity dependent potentials in the theories we are considering.

The number of Hermitian fields that appear in the Lagrangian depend on the spin and possible charge of the physical field that we are considering. For a spin-0 field with no internal flavor index (charge) we need only one Hermitian field component, i.e. $r = s = 1$. If we have a charged scalar field, then we need a pair of Hermitian fields with $r, s = 1$ and an extra flavor index $\alpha = 1, 2$. The Lagrangian will then contain two pieces corresponding to each value of α . For spin-1/2 and no charge, $r, s = 1, \dots, 4$ as in the Majorana theory. The number of hermitian fields have to be doubled to eight, $r, s = 1 \dots 4, \alpha = 1, 2$ when we are considering the Dirac theory for a spinor carrying a charge. The Hermitian fields are used here (following Schwinger) expressly for the purpose of demonstrating the spin-statistics connection. For most other purposes this choice is unnecessarily cumbersome.

3.1. Singular Lagrangians

If we require that the kinematic term of the Lagrangian be written in the form in (2) then we have to consider the possibility that K_{rs}^0 is a singular matrix. This would happen when some of the equations that we write down for the fields $\xi^{(r)}$ in the theory are not equations of motion (no time derivatives) but rather are equations of constraint. In the discussion that follows we will talk about the symmetry properties of K_{rs}^0 with respect to the indices r and s . We will also talk about the commutation or anticommutation relation of the fields ξ in terms of K_{rs}^0 and its inverse. Since we will be considering theories with primary constraints K_{rs}^0 will almost always be singular and because of this, *when we talk about the symmetry properties of K_{rs}^0 or about the inverse of K_{rs}^0 it must be understood that we are talking only about the non-singular part of the matrix.*

4. Sudarshan's observation that leads to the spin-statistics connection

The matrix K_{rs}^0 appearing in the Lagrangian in (2) is a numerical matrix that has no direct dependence on space and time coordinates. It depends on the spin labels only. It follows that if the Lagrangian has to be rotationally invariant (scalar) then the products of fields and their

derivatives that appear in the Lagrangian are themselves scalars. It will turn out that K_{rs}^0 must have specific symmetry properties with respect to the spin indices (r, s) for the Lagrangian to be nontrivial.

Obtaining the correct spin-statistics connection hinges on Sudarshan's observation [2, 3] that for the $SO(3)$ group of proper rotations (or rather its covering group $SU(2)$) in three dimensions, the representations belonging to *integral spin* have a bilinear scalar product that is *symmetric* in the indices of the factors. For example the scalar product of two real vectors is

$$(V^{(1)}, V^{(2)}) = \sum_{j,k=1,2,3} V_j^{(1)} V_k^{(2)} \delta_{jk} = \sum_{j,k=1,2,3} V_j^{(2)} V_k^{(1)} \delta_{jk} = (V^{(2)}, V^{(1)}). \quad (3)$$

On the other hand, *half-integral spin* representations have *antisymmetric* bilinear scalar products. For instance, for the spin-1/2 case the scalar product of two Hermitian spinors is

$$\begin{aligned} (\psi^{(1)}, \psi^{(2)}) &= \sum_{r,s=1}^4 \psi_r^{(1)} \psi_s^{(2)} (i\beta_{rs}) = \sum_{r,s=1}^4 \psi_s^{(1)} \psi_r^{(2)} (i\beta_{sr}) \\ &= - \sum_{r,s=1}^4 \psi_r^{(2)} \psi_s^{(1)} (i\beta_{rs}) = -(\psi^{(2)}, \psi^{(1)}), \end{aligned} \quad (4)$$

where

$$\beta_{rs} = \begin{pmatrix} 0 & \hat{\sigma}_2 \\ \hat{\sigma}_2 & 0 \end{pmatrix}$$

is an imaginary antisymmetric matrix corresponding the Majorana representation of the spinors.

The requirement that the Lagrangian be $SU(2)$ invariant (in the Hermitian field basis) means that integral spin fields appear in the Lagrangian in symmetrized scalar combinations while half integral fields appear in antisymmetrized scalar combinations. These requirements put restrictions on the symmetry properties of the invariant matrices K_{rs}^0 appearing in the Lagrangian.

To investigate the symmetry requirements on K_{rs}^0 we start from the Kinematic term of the Lagrangian in (2) i.e.

$$\mathcal{L}_{kin} = \sum_{r,s} \frac{1}{2} K_{rs}^0 (\xi^{(r)} \dot{\xi}^{(s)} - \dot{\xi}^{(r)} \xi^{(s)}). \quad (5)$$

\mathcal{L}_{kin} may be rewritten in the following form

$$\mathcal{L}_{kin} = \frac{1}{2} \sum_{r,s} \xi^{(r)} \Lambda_{rs} \dot{\xi}^{(s)} \quad ; \quad \Lambda_{rs} \equiv K_{rs}^0 (\partial_t^{(r)} - \partial_t^{(s)}).$$

For tensor fields the scalar product constructed out of $\xi^{(r)}$ and $\xi^{(s)}$ is symmetric in the indices r and s . It follows that if the Lagrangian is to be non-trivial, Λ_{rs} should also be symmetric in r and s .

$$\Lambda_{rs} = \Lambda_{sr} \quad \Rightarrow \quad K_{rs}^0 (\partial_t^{(r)} - \partial_t^{(s)}) = -K_{sr}^0 (\partial_t^{(r)} - \partial_t^{(s)}).$$

We see that because of the antisymmetry of the time derivative term, K_{rs}^0 has the opposite symmetry of Λ_{rs} and so for tensor fields, K_{rs}^0 must be an antisymmetric matrix.

For spinor fields, since the scalar product is antisymmetric, Λ_{rs} also has to be antisymmetric. Consequently for spinor fields, K_{rs}^0 must be a symmetric matrix.

Starting from the observation that for *integral spin* fields the numerical matrix K_{rs}^0 must be antisymmetric in the spin indices r and s while it must be symmetric for *half integral spin* fields due to rotational invariance of the Lagrangian; if we can now show by independent means that for *commuting fields* K_{rs}^0 must be antisymmetric and for *anticommuting fields* it must be symmetric we will obtain the proper spin-statistics connection. This is achieved using the Principle of Least Action for field quantities as formulated by Weiss and Schwinger [21, 20].

5. The principle of least action

The generator of infinitesimal transformations on the eigenstates of a complete set of commuting operators of a quantized system/field is obtained by the extended variation of the quantities contained in the action integral [20]

$$I = \int_{\sigma_2}^{\sigma_1} (dx) \mathfrak{L}[x]. \quad (6)$$

The integral is over an n -dimensional (space-time) domain with coordinates labeled by x_k ; $k = 1, \dots, n$ and bounded by the $n - 1$ dimensional surfaces σ_1 and σ_2 . In the general case σ_i are $(n - 1)$ dimensional equal time slices in the n -dimensional space-time. As we already stated, in this discussion we consider *only* theories in three space and one time dimensions. Accordingly σ_i are *three dimensional, equal time slices*. In equation (6), for \mathfrak{L} we use the specific form of the kinematic part of Lagrangian density that we wrote down in equation (2).

The Principle of Stationary Action is then formulated for operator dynamical variables as the statement that the action integral operator is unaltered by infinitesimal variations of the field quantities in the interior of the region bounded by σ_1 and σ_2 ; it being dependent only on the variation of the complete commuting set of operators attached to the bounding surfaces.

The variation of the action integral can then be shown to be decomposable into two terms;

$$\delta I = \int_{\sigma_1}^{\sigma_2} dt d^3x \delta \mathcal{L}[x] + \int d^3x (\mathfrak{L}[x, t_2] - \mathfrak{L}[x, t_1]) \delta x. \quad (7)$$

The first term is an integral over the space-time domain \mathcal{D} bounded by σ_1 and σ_2 . The second term is a “surface” term integrated only over the three spaces at times t_1 and t_2 . Setting the variation of the action in the interior of the space-time domain \mathcal{D} to zero leads to the Euler-Lagrange equations of motion for the fields while the surface variation term is treated as the generator of infinitesimal transformations on the system.

We need not go into the details of this development here but as Schwinger, pointed out, $\delta \mathfrak{L}$ should be treated carefully in the quantum case since the order of the operators that appear in \mathfrak{L} must not be altered in the course of effecting the variation. Accordingly, we recognize that the commutation properties of $\delta \xi^{(r)}$ that appear in the variation of the action based on the Lagrangian density in equation (2) are involved in the consequences of the extended variation of the action integral. In other words we must, at this point, make the explicit assumption that the commutation properties of $\delta \xi$ with respect to $\xi^{(r)}$ and the structure of the Lagrange function must be connected in a consistent fashion.

We now choose to simplify matters as much as possible and as before focus on \mathfrak{L}_{kin} . We are interested in the variation of the action that is brought about by the variation in the field quantities ξ themselves. We will not therefore consider the variation of the action integral operator that is brought out by a change in the coordinates x or time t . The variations that we consider here have the form

$$\xi_k \rightarrow \xi_k + \delta \xi_k, \quad \dot{\xi}_k \rightarrow \dot{\xi}_k + \frac{d(\delta \xi_k)}{dt}. \quad (8)$$

The ensuing variation in the action is then given by

$$\delta I_{\delta \xi} = \frac{1}{2} \int dt d^3x \left(\frac{\partial \mathfrak{L}}{\partial \xi^{(r)}} \delta \xi^{(r)} + \frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \delta \dot{\xi}^{(r)} \right). \quad (9)$$

The integral is over a three dimensional space and one dimensional time domain. Also, the above equation must again be understood in a symbolic sense with the exact position of the

variation $\delta\xi$ in each term to be fixed later in an appropriate fashion. Using the identity,

$$\frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \delta \dot{\xi}^{(r)} = \frac{\partial}{\partial t} \left(\frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \delta \xi^{(r)} \right) - \frac{\partial}{\partial t} \left(\frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \right) \delta \xi^{(r)},$$

and using Gauss' theorem we rewrite the variation in the action integral as

$$\delta I_{\delta\xi} = \frac{1}{2} \int dt d^3x \left\{ \frac{\partial \mathfrak{L}}{\partial \xi^{(r)}} - \frac{\partial}{\partial t} \left(\frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \right) \right\} \delta \xi^{(r)} + \frac{1}{2} \int_{\sigma} d^3x \frac{\partial \mathfrak{L}}{\partial \dot{\xi}^{(r)}} \delta \xi^{(r)}. \quad (10)$$

Setting the first term (variation in the interior of the domain \mathcal{D} bounded by σ) to zero we obtain the Euler-Lagrange equations for the hermitian fields $\xi^{(r)}$. It is the surface variation terms that are of interest to us as the generators of infinitesimal transformations on the field quantities themselves. For the specific form for \mathfrak{L}_{kin} that we are considering in equation (5), the (surface) variation term is given by

$$\delta I_{\delta\xi} = \frac{1}{2} \int_{\sigma} d^3x \sum_{r,s} K_{rs}^0 (\xi^{(r)} \delta \xi^{(s)} - \delta \xi^{(r)} \xi^{(s)}). \quad (11)$$

Since $\delta I_{\delta\xi}$ is the generator of infinitesimal transformations of the field quantities ξ attached to the surface σ , using Eq. (1) we obtain

$$\frac{1}{2} \int_{\sigma} d^3x \left[\xi^{(n)}, \sum_{r,s} K_{rs}^0 (\xi^{(r)} \delta \xi^{(s)} - \delta \xi^{(r)} \xi^{(s)}) \right] = i\hbar \delta \xi^{(n)}. \quad (12)$$

Expanding out the commutator we obtain

$$\frac{1}{2} \int_{\sigma} d^3x \sum_{r,s} K_{rs}^0 (\xi^{(n)} \xi^{(r)} \delta \xi^{(s)} - \xi^{(n)} \delta \xi^{(r)} \xi^{(s)} - \xi^{(r)} \delta \xi^{(s)} \xi^{(n)} + \delta \xi^{(r)} \xi^{(s)} \xi^{(n)}) = i\hbar \delta \xi^{(n)}. \quad (13)$$

Now we are in a position to assume that the fields appearing in equation (13) are either commuting fields or anticommuting fields and investigate what restrictions (if any) each assumption places on the matrix K_{rs}^0 .

5.1. Commuting fields

Let us assume now that the fields ξ are bosonic fields and we consider variations in them. The significant step is that we assume $\delta\xi$ commutes with everything. Using this assumption we can rewrite the left hand side of equation (13) as

$$\frac{1}{2} \int_{\sigma} d^3x \sum_{r,s} K_{rs}^0 \{ (\xi^{(n)} \xi^{(r)} \delta \xi^{(s)} - \xi^{(r)} \xi^{(n)} \delta \xi^{(s)}) - (\xi^{(n)} \xi^{(s)} \delta \xi^{(r)} - \xi^{(s)} \xi^{(n)} \delta \xi^{(r)}) \} \quad (14)$$

Putting in the space coordinates \mathbf{x} and \mathbf{y} of the fields explicitly we can rewrite the previous expression in terms of commutation relations of the fields as

$$\frac{1}{2} \int_{\sigma} d^3x \sum_{r,s} K_{rs}^0 \{ [\xi^{(n)}(\mathbf{y}), \xi^{(r)}(\mathbf{x})] \delta \xi^{(s)}(\mathbf{x}) - [\xi^{(n)}(\mathbf{y}), \xi^{(s)}(\mathbf{x})] \delta \xi^{(r)}(\mathbf{x}) \}. \quad (15)$$

Exchanging the indices i and j in the second term we can rewrite the expression in equation (15) as

$$[\xi^{(n)}, \delta I_{\delta\xi}] = \int_{\sigma} d^3x \sum_s \delta \xi^{(s)}(\mathbf{x}) \left[\xi^{(n)}(\mathbf{y}), \frac{1}{2} \sum_r (K_{rs}^0 - K_{sr}^0) \xi^{(r)}(\mathbf{x}) \right] \quad (16)$$

From equation (16) we see explicitly that $[\xi^{(n)}, \delta I_{\delta\xi}] = 0$ if K_{rs}^0 is symmetric. So to satisfy the condition that the surface variation of the action is the generator of infinitesimal transformations we need K_{rs}^0 to be an *antisymmetric* matrix if the fields $\xi^{(r)}$ (and $\delta\xi^{(r)}$) are bosonic commuting fields. The possibility that K_{rs}^0 be neither symmetric nor antisymmetric ($K_{rs}^0 \neq \pm K_{sr}^0$) in the spin indices is already excluded by the rotational invariance of the Lagrangian density that we require.

5.2. Anticommuting fields

If we assume fermionic anticommutation relations instead of commutation relations for the fields ξ and assume that $\delta\xi$ *anticommutes* with everything, the analogue of equation (14) that we obtain from (13) is

$$\frac{1}{2} \int_{\sigma} d^3x \sum_{r,s} K_{rs}^0 \{(\xi^{(n)}\xi^{(r)}\delta\xi^{(s)} + \xi^{(r)}\xi^{(n)}\delta\xi^{(s)}) + (\xi^{(n)}\xi^{(s)}\delta\xi^{(r)} + \xi^{(s)}\xi^{(n)}\delta\xi^{(r)})\} \quad (17)$$

This can be simplified as before to

$$[\xi^{(n)}, \delta I_{\delta\xi}] = \int_{\sigma} d^3x \sum_s \delta\xi^{(s)}(\mathbf{x}) \left\{ \xi^{(n)}(\mathbf{y}), \frac{1}{2} \sum_r (K_{rs}^0 + K_{sr}^0)\xi^{(r)}(\mathbf{x}) \right\} \quad (18)$$

In this case we see that $[\xi^{(n)}, \delta I_{\delta\xi}] = 0$ if K_{rs}^0 is antisymmetric. Thus for anticommuting fermionic fields we require K_{rs}^0 to be a *symmetric* matrix to be consistent with the dynamical principle.

6. The spin-statistics connection

We have seen that the rotational invariance of the kinematic part of the Lagrangian density requires that the matrix K_{rs}^0 be antisymmetric for integral spin fields while it be symmetric for half-integral fields. The Schwinger Principle of Least Action on the other hand requires that K_{rs}^0 be antisymmetric for commuting (bosonic) fields and symmetric for anticommuting (fermionic) fields. This leads us to the conclusion that *fields with integral spin must be bosonic while fields with half-integral spin must be fermionic*. This is the spin-statistics connection.

7. Discussion

We use hermitian fields in \mathfrak{L}_{kin} in our proof of the spin-statistics connection. In hermitian fields, the creation and annihilation parts always appear together. For relativistic fields, the requirement of locality assures us that the two parts always appear together. In the non-relativistic case, imposing hermiticity on the fields, automatically realizes the *Kirchoff's principle*. When we study the thermodynamics of radiation in equilibrium, we assume that there is a continual emission and absorption of the radiation by matter. Furthermore we know that the emissivity and absorptivity of matter are proportional to each other: this is Kirchoff's Principle. In terms of ordinary quantum mechanics what this means is that when we couple harmonic oscillator degrees of freedom, the coupling is in terms of the coordinate $q \sim (a + a^\dagger)/2$ rather than coupling the creation and annihilation operators separately and independently. In an analogous manner we realize Kirchoff principle for the non-relativistic fields by. The requirement that the probabilities of creating and destroying the quanta of the fields that we are considering are proportional to each other ensures that the creation and annihilation fields do not appear independently in the Lagrangians that we are considering.

In our discussion of the Principle of Least Action for quantum mechanical operators and fields in section 5, we considered the variation of the action when the fields ξ_r are varied by $\delta\xi_r$. To get commutation or anticommutation relations for the fields and to obtain the correct spin-statistics connection we made the choice that $\delta\xi_r$ commutes with every quantity that appears

in the variation of the action when ξ_r denote commuting fields and that $\delta\xi_r$ anticommutes with everything if ξ_r are anticommuting quantities. This is required if we want to keep the varied fields $\xi'_r = \xi_r + \delta\xi_r$ such that they have *exactly the same commutation or anticommutation properties as ξ_r* .

Choosing the arbitrary variation $\delta\xi_r$ to be a quantity that commutes with everything is easy enough because all one has to do is to make sure that $\delta\xi$ is a c-number. On the other hand, choosing $\delta\xi_r$ such that it anticommutes with all the field quantities that appear in $\delta I_{\delta\xi}$ is not so straightforward. For an even number of anticommuting fields, ξ_r , one possible choice to make $\delta\xi_r$ proportional to the product of all the fields, i.e

$$\delta\xi_r \sim \epsilon_r \prod_{i=1}^{2n} \xi_i \quad ; \quad r = 1, 2, \dots, 2n.$$

$\delta\xi_r$ will then anticommute with all the quantities that appear in the variation of the action [25].

Restricting $\delta\xi_r$ to be either commuting or anticommuting with everything is essential for the canonical quantization of commuting or anticommuting fields [26, 27]. But it is possible to make the choice of $\delta\xi_r$ more general. If we consider the fields ξ_r defined on the equal time slice σ as forming a complete set of operators on σ , a natural generalization of $\delta\xi_r$ is to choose it as a linear combination of ξ_r ,

$$\delta\xi_r(\mathbf{x}) = \sum_s \epsilon_{rs} \xi_s(\mathbf{x}) \tag{19}$$

where ϵ_{rs} are infinitesimal c-number coefficients. The basic commutation relation in (12) can be rewritten as

$$\frac{1}{2} \int_{\sigma} d^3x \left[\xi_n, \sum_{k,r,s} K_{rs}^0 (\xi_r \epsilon_{sk} \xi_k - \epsilon_{rk} \xi_k \xi_s) \right] = i\hbar \sum_k \epsilon_{nk} \xi_k. \tag{20}$$

Without working out the details we can immediately see that eq. (20) will lead to *trilinear* commutation or anticommutation relations among ξ_r . These generalized commutation and anticommutation relations, due to Wigner and Green [28, 29], lead to reducible para-Bose and para-Fermi systems.

The arguments leading to the spin-statistics connection rely on the symmetry, or antisymmetry, that is required independently by $SU(2)$ invariance of the Lagrangian and by the Principle of Least Action, of the numerical matrix K_{rs}^0 . It might seem possible that the introduction of internal symmetry indices (flavors) on the fields ξ_r can change the symmetry requirements on K_{rs}^0 . It turns out if we try to do this we will end up with negative norm states in the theory [24].

8. conclusion

We have shown that for the general class of fields that can be canonically quantized, Sudarshan's ideas lead to a proof of the spin-statistics connection that does not depend on relativistic arguments. All that is required is the $SU(2)$ invariance of the kinematic part of the Lagrangian. The four requirements on \mathfrak{L}_{kin} listed in Sec. 3 for the proof to work are quite reasonable and justified especially when we are dealing with non-relativistic systems. It is important to note that the requirements are placed just on the kinematic part. This means that we obtain the spin-statistics connection using just the properties of the system and not using the behavior of the system under any dynamical process.

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