Quantum computing with spin networks

Andrew Landahl[*], Matthias Christandl[†], Nilanjana Datta[**] and Artur Ekert[†]

[*]*Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA*
[†]*Centre for Quantum Computation, Centre for Mathematical Sciences, DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*
[**]*Statistical Laboratory, Centre for Mathematical Sciences, DAMTP, University of Cambridge, Wilberforce Road, Cambridge CB3 0WA, UK*

Abstract. We present a class of spin networks that act as perfect quantum wires—quantum states transfer across these networks with unit fidelity in a time proportional to the network's size. No couplings need to be switched on and off during this process. We prove that N-spin hypercube networks with equal-strength nearest-neighbor Heisenberg and XY couplings have a maximal perfect communication distance of $2\log_3 N$. We show how to engineer coupling strengths in linear Heisenberg and XY spin chains to efficiently achieve perfect state transfer over arbitrarily long distances. Finally, we show how to augment these linear spin chain networks with more complex, but static, interactions to achieve universal quantum computation.

Quantum circuits are built from quantum wires and quantum gates. How these components are technologically realized depends very much on the application. For example, wires might take the form of "flying qubit" photons for long-distance quantum communication whereas they might take the form of "bucket brigade" optical lattice modulations for quantum memory. For applications in which qubits are packed at high density, it is worthwhile to investigate the extent to which direct qubit-qubit interactions can be harnessed to technologically realize quantum wires and quantum gates.

A useful model to consider in this context is a spin network—a collection of N qubits with nearest-neighbor interactions describable by a graph. Recently, equal-strength Heisenberg and XY linear spin chain networks have been shown to act as quantum wires with high, but non-unit fidelity [1]. In this paper, we examine spin networks that can serve as *perfect*, i.e., unit fidelity, quantum wires. We also show that these perfect quantum wires can be augmented to perform perfect quantum gates.

A spin network is described by a graph G with vertices $V(G)$ representing spins and edges $E(G)$ representing interactions between them. The *adjacency matrix* corresponding to this graph is

$$A_{ij}(G) := \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We label the two vertices in $V(G)$ between which we wish to consider perfect state transfer A and B.

Consider spin networks whose interactions are time-independent Hamiltonians. In the subsequent, we consider specifically an XY coupling

$$H_G = \frac{1}{2} \sum_{(i,j)\in E(G)} \left[\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y \right], \tag{2}$$

where σ_i^x, σ_i^y and σ_i^z are the Pauli matrices acting on the ith qubit; later we consider other interactions. Using the (nonstandard) identification between spin states and qubits $|0\rangle \equiv |\downarrow\rangle$ and $|1\rangle \equiv |\uparrow\rangle$, we denote the ground state of an XY spin network by $|\underline{0}\rangle = |0_A 00\cdots 00_B\rangle$.

Although the Hilbert space $\mathscr{H}_G$ of the N-spin network has dimension 2^N, state transfer dynamics occurs completely within the N-dimensional subspace $\mathscr{S}_G$ spanned by the basis vectors $|n\rangle$ with $n = 1,\ldots,N$, corresponding to spin configurations in which all spins are 'down' apart from just one spin at the vertex n which is 'up'. This is because H_G commutes with the total spin operator in the z-direction, forcing total spin in the z-direction to be conserved.

When restricted to this subspace, $H_G|_{\mathscr{S}_G}$ is represented by an $N \times N$ matrix that is *identical* to the adjacency matrix $A(G)$, Eq. (1), of the underlying graph G. Because of this, the time evolution of the network in the $\mathscr{S}_G$ subspace is a continuous-time quantum walk on G [2].

The question we are interested in is: When will the quantum walk propagate from A to B with unit fidelity? To answer this, we need to compute the probability amplitude

$$F(t) = \langle N | e^{-itH_G} | 1 \rangle . \tag{3}$$

Perfect state transfer is obtained for times t for which $|F(t)| = 1$.

For the XY linear spin chain, one can compute $F(t)$ explicitly by diagonalizing the Hamiltonian or the corresponding adjacency matrix. The eigenstates and the corresponding eigenvalues are

$$|\tilde{k}\rangle = \sqrt{\frac{2}{N+1}} \sum_{n=1}^{N} \sin\left(\frac{\pi k n}{N+1}\right) |n\rangle , \qquad E_k = -2\cos\frac{k\pi}{N+1}, \tag{4}$$

$$\tag{5}$$

with $k = 1, \ldots, N$. Thus

$$F(t) = \frac{2}{N+1} \sum_{k=1}^{N} \sin\left(\frac{\pi k}{N+1}\right) \sin\left(\frac{\pi k N}{N+1}\right) e^{-iE_k t}. \tag{6}$$

Perfect state transfer from one end of the chain to another is possible *only* for $N = 2$ and $N = 3$, with $F(t) = -i\sin t$ and $F(t) = -\sin^2(t/\sqrt{2})$ respectively. For perfect state transfer in a graph G, it is necessary that the ratios of the differences of eigenvalues of the related adjacency matrix $A(G)$ are rational numbers. The absence of perfect state transfer for $N \geq 4$ can be proved by showing explicitly that the above condition is not satisfied.

A chain of two or three qubits can serve as basic building blocks for networks that can perfectly transfer a quantum state over longer distances. This can be achieved by building networks which are multiple Cartesian products of either of the two simple chains.

In general the Cartesian product of two graphs $G := \{V(G), E(G)\}$ and $H := \{V(H), E(H)\}$ is a graph $G \times H$ whose vertex set is $V(G) \times V(H)$ and two of its vertices (g,h) and (g',h') are adjacent if and only if one of the following hold: (i) $g = g'$ and $\{h,h'\} \in E(H)$; (ii) $h = h'$ and $\{g,g'\} \in E(G)$. If $|\tilde{k}\rangle$ is an eigenvector of $A(G)$ corresponding to eigenvalue E_k and $|\tilde{l}\rangle$ is an eigenvector of $A(H)$ corresponding to eigenvalue E_l then $|\tilde{k}\rangle \otimes |\tilde{l}\rangle$ is an eigenvector of $A(G \times H)$ corresponding to eigenvalue $E_k + E_l$. This is because

$$A(G \times H) = A(G) \otimes \mathbb{1}_{V(H)} + \mathbb{1}_{V(G)} \otimes A(H), \tag{7}$$

where $\mathbb{1}_{V(H)}$ is the $|V(H)| \times |V(H)|$ identity matrix (see e.g. [3]).

Now, consider a graph G^d which is a d-fold Cartesian product of graph G. The propagator between the two antipodal vertices in G^d, namely $A = (1, \ldots, 1)$ and $B = (N, \ldots, N)$, is simply

$$F_{G^d}(t) = [F_G(t)]^d . \tag{8}$$

The d-fold Cartesian product of a one-link chain (two qubits) and a two-link chain (three qubits) lead to one-link and two-link hypercubes with $|F(t)|$ given, respectively, by

$$\sin^d t \quad \text{and} \quad \sin^{2d}\left(t/\sqrt{2}\right). \tag{9}$$

Any quantum state can be perfectly transferred between the two antipodes of the one-link and two-link hypercubes of any dimensions in constant time $t = \pi/2$ and $t = \pi/\sqrt{2}$ respectively.

Thus we have shown that for a two-link hypercube of N sites, the maximum distance of perfect quantum communication is $2\log_3 N$.

An improvement to perfect quantum communication distance is possible if one allows different, but fixed, couplings. To see this, relabel the basis vectors of an N-spin chain as $|m\rangle$, where $m = -\frac{1}{2}(N-1) + n - 1$ and let the interaction Hamiltonian be a modified version of (2),

$$H_G = \sum_{(n,n+1) \in E(G)} \frac{J_n}{2} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y \right]. \tag{10}$$

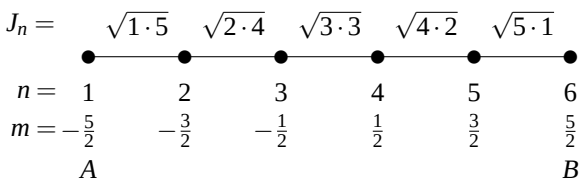

FIGURE 1. Couplings J_n that admit perfect state transfer from A to B in a 6-qubit chain. Eigenvalues m of the equivalent spin-$\frac{5}{2}$ particle are also shown.

When restricted to the subspace $\mathscr{S}_G$, the Hamiltonian H_G takes the form

$$\begin{pmatrix} 0 & J_1 & 0 & \cdots & 0 \\ J_1 & 0 & J_2 & \cdots & 0 \\ 0 & J_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & J_{N-1} \\ 0 & 0 & 0 & J_{N-1} & 0 \end{pmatrix}, \tag{11}$$

which is identical to the Hamiltonian H of a fictitious spin $(N-1)/2$ particle when the couplings are engineered to the values

$$J_n = \frac{\lambda}{2}\sqrt{n(N-n)}; \tag{12}$$

an example for $N = 6$ is depicted in Fig. 1. In this case, $H = \lambda S_x$, where S_x is the fictitious spin's angular momentum operator and λ is some constant. The evolution $U(t) = \exp(-i\lambda t\, S_x)$ of the network represents a rotation of this fictitious particle. The matrix elements $\langle n'|\,U(t)\,|n\rangle$ of this rotation matrix are well-known and in particular the probability amplitude for state transfer is

$$F(t) = \langle N|\,U(t)\,|1\rangle = \sin^{N-1}(\lambda t/2)\,. \tag{13}$$

Thus perfect transfer of a quantum state between the two antipodes A and B is obtained in a constant time $t = \pi/\lambda$, or when the amount of energy available is bounded so that the interaction strength in the center of the chain is constant, $\lambda \sim 1/N$ and state transfer takes a time $t \sim N$ to occur.

Each such engineered qubit chain can be viewed as a projection from a graph having identical qubit couplings. In fact, there is an entire family of such graphs $\mathscr{G}$ that project to this chain. Motivated by the 'column method' of [4], we define $\mathscr{G}$ as the set of graphs whose vertices can be partitioned into N columns G_n of size $|G_n| = \binom{N-1}{n-1}$ that satisfy the following two conditions for $n = 1,\ldots,N$: (*i*) each vertex in column n is connected to $N-n$ vertices in column $n+1$, and (*ii*) each vertex in column $n+1$ is connected to n vertices in column n. An important example of a graph in $\mathscr{G}$ is the one-link hypercube, where columns are defined as the set of vertices reachable in n links. The evolution of a state at A (the first column) under H_G (eq. (2)) remains in the column space $\mathscr{H}_{col} \subseteq \mathscr{H}_G$, spanned by

$$|\,\mathrm{col}\; n\rangle = \frac{1}{\sqrt{|G_n|}}\sum_{m=1}^{|G_n|}|\,G_{n,m}\rangle \tag{14}$$

where $G_{n,m}$ labels the vertices in G_n. Hence, we restrict our attention to $\mathscr{H}_{col}$ in which the matrix elements of H_G are given by

$$J_n = \langle \mathrm{col}\; n\,|\,H_G\,|\,\mathrm{col}\; n+1\rangle = \sqrt{n(N-n)}, \tag{15}$$

the same as in the engineered chain.

In our analysis we have focused on qubits coupled with the XY interaction. The choice of this interaction was dictated by its simple connection with the adjacency matrix. We should add, however, that our considerations remain valid if we choose the Heisenberg interaction and compensate for the diagonal elements in the S_G subspace. For example, the Heisenberg model with local magnetic fields,

$$\tfrac{1}{2}\sum_{j=1}^{N-1} J_j\;\boldsymbol{\sigma}_j\cdot\boldsymbol{\sigma}_{j+1} + \sum_{j=1}^{N} B_j\sigma_j^{(z)}, \tag{16}$$

with $B_n = \frac{1}{2}(J_{n-1} + J_n) - \frac{1}{2(N-2)} \sum_{k=1}^{N-1} J_k$, gives exactly the same state transfer dynamics as the XY model.

Our analysis is not restricted to pure states; the method presented here works equally well for mixed states. It can also be used to transfer or to distribute quantum entanglement.

Given the preceding construction for a perfect quantum wire, one may similarly implement a perfect quantum gate, and indeed a perfect quantum computation using spin networks. Given an N-qubit quantum circuit $U = U_T \cdots U_1$, consider the following time-independent Hamiltonian:

$$H = \sum_{t=1}^{T} J_t \left(U_t \otimes |t\rangle \langle t-1| + U_t^\dagger |t-1\rangle \langle t| \right) \tag{17}$$

Feynman proved [5] that when $J_t = 1$, the spectrum of H is the same as that of the hopping Hamiltonian one obtains when each $U_t = \mathbb{1}$. Indeed, $\langle \psi_t | H | \psi_{t+1} \rangle = 1$ between the states $|\psi_t\rangle = U_t \cdots U_1 |0_1 \ldots 0_N\rangle |t\rangle$, indicating that H is nothing more than the discretized version of a free particle in one dimension. If this system were left to freely evolve and were measured at time $T/2$, the probability that the computation succeeds is

$$\left| \langle \psi_T | e^{-iHT/2} | \psi_0 \rangle \right|^2 = O(T^{-2/3}). \tag{18}$$

Rather than repeat $O(T^{2/3})$ times so that the probability of success is close to one, Feynman suggested preparing a wavepacket to ballistically propagate through the network so that the computation succeeds with high probability in a single shot. Using our perfect wire construction, we can do even better. If we choose the J_t of Eq. (17) to vary as Eq. (12), then the computation will coherently evolve to its final configuration with unit fidelity in a time proportional to T.

In conclusion, we have proven that perfect quantum state transfer between antipodal points of one-link and two-link N-spin hypercube networks is possible in a time proportional to N and that perfect quantum state transfer between antipodal points of k-link hypercubes for $k \geq 3$ is impossible. In addition, we have shown that a quantum state can be transferred perfectly over a chain of *any* length as long as one can pre-engineer inter-qubit interactions. This construction can be combined with Feynman's spin-network Hamiltonian (17) to generate any quantum computation with unit fidelity entirely within a spin network. These networks are especially appealing as they require no dynamical control, unlike many other quantum information processing proposals.

ACKNOWLEDGMENTS

This work was supported by the CMI grant, A*Star Grant No. 012-104-0040. MC was supported by a DAAD Doktorandenstipendium and the EPSRC. ND would like to thank N. Cooper and A. Bovier for helpful discussions. AL was supported by a Hewlett-Packard Fellowship and would like to thank A. Childs, S. Gutmann, E. Farhi, and J. Goldstone for helpful discussions.

REFERENCES

1. Bose, S., *Phys. Rev. Lett.*, **91**, 207901 (2003).
2. Farhi, E., and Gutmann, S., *Phys. Rev. A*, **57**, 2403–2406 (1998).
3. Wilson, R. J., and Beineke, L. W., editors, *Selected Topics in Graph Theory*, Academic Press, London, 1979.
4. Childs, A. M., Cleve, R., Deotto, E., Farhi, E., Gutmann, S., and Spielman, D. A., "Exponential algorithmic speedup by a quantum walk," in *Proceedings of the Thirty-Fifth Annual ACM Symposium on the Theory of Computing*, edited by L. L. Larmore and M. X. Goemans, ACM, ACM Press, New York, 9–11 Jun. 2003, San Diego, CA, USA, 2003, pp. 59–68, ISBN 1-58113-674-9.
5. Feynman, R. P., *Found. Phys.*, **16**, 507–531 (1986).