UNM Physics 262, Fall 2006
SAMPLE Midterm Exam 2: Relativity

Name and/or CPS number: Dr. Landahl - Solution Key

Show all your work for full credit. Remember that quantities have units and vectors have components (or magnitude and direction). **ASK** if anything seems unclear.

CALCULATORS AND CELL PHONES ARE PROHIBITED.
USE OF THESE WILL RESULT IN A ZERO FOR THE EXAM.
Keep any factors of π, e, $\sqrt{2}$, etc. in your answers.

You may use a single 8.5" $\times$ 11" paper containing notes you have prepared ahead of time to assist you.

Apportion your time sensibly. Spend about 10–12 minutes per problem.

Please put a box around your final answers.

Problem 1: 25/25

Problem 2: 25/25

Problem 3: 25/25

Problem 4: 25/25

100/100

1. Short answer [25 points]

The questions below should be answered with no more than five lines of text and no calculations. Please be brief and to the point.

[5] *a*) Define an inertial reference frame. Give an example of a *non*-inertial reference frame.

IRF: A reference frame in which Newton's 3rd Law (the law of inertia) holds. A reference frame is a regular lattice of synchronized clocks.

NIRF examples: Regener Hall 103, an accelerating car, a Merry-go-round, etc.

[5] *b*) What does it mean for two spacetime events to be spacelike separated? Timelike separated? Lightlike separated?

spacelike separated: neither event can cause or be caused by the other. The <u>space</u> ordering of these events is the same in all IRFs.

timelike " : one event could have caused the other. The <u>time</u> ordering of these events is the same in all IRFs

lightlike " : A causal connection between these events can only be by a massless particle (like light) The <u>time and space</u> ordering of these events is the same in all IRFs.

[5] *c*) In words, what are the important differences between the Doppler shift derived using special relativity and the Doppler shift derived using Galilean relativity?

The Galilean Doppler effect depends on the speed the objects have with respect to the sound medium (e.g., air). Light has no such medium —— only the relative velocity matters. Also, the SR D.E. occurs for <u>transverse</u> and longitudinal motion relative to the light signals, whereas the Galilean DE is longitudinal only.

[5] *d*) Evaluate the following statement: "As the speed of a moving clock approaches the speed of light, time stops."

In the frame of such a clock, time flows at the same rate as when it is at rest. In the frame relative to which the clock is moving, the clock appears to move more and more slowly, and time itself in the clock's frame approaches stopping. Time is never measured to stop entirely for a massive object, however, because only massless objects can travel at the speed of light

[5] *e*) State the postulates upon which Einstein based special relativity.

① The Principle of Relativity: The laws of physics are the same in all IRFs.

② The speed of light is constant in all IRFs.

2. Spacetime geometry [25 points]

By chance, the twins Alice and Bob have enrolled in Physics 262. Inspired by the course, Bob has devised a clever strategy for preserving his youth. (Alice is more sensible.) Bob has attached himself to a spring so that his x-coordinate oscillates according to

$$x(t) = x_0 \sin \omega t.$$

The product $x_0\omega$ has units of velocity, so define $\beta_0 \equiv x_0\omega/c$. Excited by the prospects of preserving his youth, but wary of testing the limits of his mechanics, Bob sets up the spring so that $\beta_0 \ll 1$, and begins to oscillate. Alice watches, rolling her eyes, from her inertial reference frame.

[8] *a*) Write down an expression relating the differential passage of time in Alice's IRF (dt) to the differential passage of Bob's proper time ($d\tau$). Simplify the relation to the point where β_0 is the only variable in the expression.

$$(c\,d\tau)^2 = (c\,dt)^2 - (dx)^2 \qquad \text{[Invariance of interval]}$$

$$dx = (x_0\omega \cos\omega t)\,dt \qquad \text{[Calculus]}$$

$$d\tau^2 = dt^2 - \left[\left(\frac{x_0\omega}{c}\right)^2 \cos^2\omega t\right] dt^2$$

$$= dt^2\left(1 - \beta_0^2 \cos^2\omega t\right)$$

$$\boxed{d\tau = \sqrt{1-\beta_0^2\cos^2\omega t}\;dt}$$

[9] *b*) Integrate this expression to find how much younger Bob is than Alice *after each oscillation*, that is, each time Bob returns to $x = 0$. [Hint: Use $\beta_0 \ll 1$ and the binomial expansion to make the calculation possible.]

τ_B = Bobs age after 1 period of oscillation.

T = period of oscillation in A's frame, $= \frac{2\pi}{\omega}$

$$\int_0^{\tau_B} d\tau = \int_{t=0}^{T} \sqrt{1-\beta_0^2\cos^2\omega t}\, dt$$

Binomial expansion: $(1-\beta_0^2\cos^2\omega t)^{1/2} \approx 1 - \frac{1}{2}\beta_0^2\cos^2\omega t$

$$\tau_B = \int_{t=0}^{2\pi/\omega} \left(1 - \tfrac{1}{2}\beta_0^2\cos^2\omega t\right) dt$$

$$= \frac{2\pi}{\omega} - \frac{1}{2}\beta_0^2 \underbrace{\int_{t=0}^{2\pi/\omega} \cos^2\omega t\, dt}_{= 1/2\, T}$$

(integral of $\cos^2$ over a period is half the period)

$$= T - \frac{\beta_0^2}{4}T = \frac{2\pi}{\omega}\left(1 - \beta_0^2/4\right)$$

$\Rightarrow$ Bob is younger by $\boxed{\frac{\pi\beta_0^2}{2\omega}}$ every period of oscillation.

[8] *c*) For $\beta_0 = 1/4$, how long would Bob have to oscillate to be one year younger than Alice?

After n periods of oscillation, Bob is $\frac{n\beta_0^2}{4}T$ younger.

If we set this to 1 year with $\beta_0^2 = 1/4$, we have

$$(nT)\frac{\beta_0^2}{4} = 1\,\text{yr}$$

$$nT = \frac{4\,\text{yr}}{\beta_0^2} = 16 \cdot 4\,\text{yr} = \boxed{64\,\text{yr}}$$

What a boring existence!

3. Relativistic kinematics [25 points]

In the distant future, Renaissance festivals have not gone away but instead have become more extreme. Two future "knights" in such a festival will engage in a joust. To ensure fairness, each rides an identical robotic horse that can travel at a maximum speed of $c/3$ relative to the Earth. Each knight is also equipped with an identical 1 m long lance. The favored knight goes by the moniker "The Green Knight," because he wears a stylized green costume ($\lambda = 500$ nm). The challenger knight wears no special decoration. The two knights approach one another at maximum speed with their lances directed towards one another.

	Red	Orange	Yellow	Green	Blue	Indigo	Violet
λ (nm)	650	600	575	500	475	450	400

[6] *a*) At what speed does the challenger measure the Green Knight to be approaching?

Velocity transformation: $\dfrac{c/3 + c/3}{1 + (\frac{c}{3})^2 \frac{1}{c^2}} = \dfrac{2c/3}{1 + 1/9} = \dfrac{6c}{10} = \boxed{\frac{3}{5}c}$

[6] *b*) What length do the spectators (at rest with respect to the ground) measure the Green Knight's lance to be?

Length contraction: $l = l'/\gamma$. $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{1}{\sqrt{1 - \frac{c^2}{9}\cdot\frac{1}{c^2}}}$

$\gamma = \dfrac{1}{\sqrt{\frac{9-1}{9}}} = \sqrt{\frac{9}{8}} = \dfrac{3}{2\sqrt{2}}$. $\Rightarrow$ $\boxed{l = \dfrac{2\sqrt{2}}{3} \text{ meters long}}$

[6] *c*) What length does the challenger measure the Green Knight's lance to be?

Length contraction: $l = l'/\gamma$, $\gamma = \dfrac{1}{\sqrt{1 - (\frac{3}{5})^2}} = \dfrac{1}{\sqrt{\frac{5^2 - 3^2}{5^2}}} = \sqrt{\dfrac{25}{25-9}}$

$\gamma = \dfrac{5}{4}$ $\Rightarrow$ $\boxed{l = \frac{4}{5} \text{ meters long}}$

[6] *d*) What color does the challenger measure the Green Knight's costume to be? (Use the table above to convert wavelength to color.)

Doppler shift (blueshift): $f = \sqrt{\dfrac{1+v/c}{1-v/c}}\, f_0 \iff \lambda = \sqrt{\dfrac{1-v/c}{1+v/c}}\, \lambda_0$

$\sqrt{\dfrac{1-v/c}{1+v/c}} = \sqrt{\dfrac{1-3/5}{1+3/5}} = \sqrt{\dfrac{5-3}{5+3}} = \sqrt{\dfrac{2}{8}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$. $\lambda = \frac{1}{2}(500 \text{ nm}) = \boxed{250 \text{ nm}}$

off the chart in the Blue direction.

[1] *e*) Would the spectators (who see the knights at a side view) measure the Green Knight's costume to be redshifted, blueshifted, or unshifted?

Transverse DE: $f = f_0/\gamma$ $\gamma > 1 \Rightarrow$ frequency measured to be smaller $\Rightarrow$ wavelength measured to be longer $\Rightarrow$ $\boxed{\text{redshifted}}$

4. Relativistic Dynamics [25 points]

A particle is measured in a certain inertial reference frame to have a total energy of 5 GeV and a momentum of 3 GeV/c (*i.e.*, cp, which has the dimensions of energy, is equal to 3 GeV).

[6] *a*) What is the mass of the particle, in GeV/c^2?

$$\vec{p}\cdot\vec{p} = m^2c^2 \Rightarrow E^2 - p^2c^2 = m^2c^4$$

$$m^2 = \frac{E^2}{c^4} - \frac{p^2}{c^2} = 25\,\frac{\text{GeV}^2}{c^4} - \frac{9\,\text{GeV}^2}{c^4} = 16\,\frac{\text{GeV}^2}{c^4} \Rightarrow \boxed{m = 4\ \text{GeV}/c^2}$$

[6] *b*) What is the velocity of the particle, v/c?

$$E = \gamma mc^2 \Rightarrow \gamma = \frac{5\ \text{GeV}}{4\ \text{GeV}} = 5/4 = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$1 - \frac{v^2}{c^2} = \frac{16}{25} \Rightarrow \frac{v}{c} = \sqrt{1-\frac{16}{25}} = \sqrt{\frac{25-16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}. \quad \boxed{\frac{v}{c} = 3/5}$$

[6] *c*) What is the energy of the particle in an IRF in which the particle's momentum is 4 GeV/c?

$$E^2 - p^2c^2 = m^2c^4 \qquad E = \sqrt{m^2c^4 + p^2c^2}$$

$$E = \sqrt{(4\,\text{GeV}/c^2)^2c^4 + (4\,\text{GeV}/c)^2c^2} = \sqrt{32}\ \text{GeV} = \boxed{4\sqrt{2}\ \text{GeV}}$$

[6] *d*) What is the kinetic energy of the particle in this new IRF?

$$KE = mc^2(\gamma - 1) = E - mc^2 = 4\sqrt{2}\ \text{GeV} - 4\ \text{GeV}$$

$$= \boxed{4(\sqrt{2}-1)\ \text{GeV}}$$

[1] *e*) What is the maximum momentum this particle can have, according to the limits set by special relativity?

SR places no upper limit on momentum.