UNM Physics 262, Problem Set 10, Fall 2006

Instructor: Dr. Landahl
Issued: November 8, 2006
Due: Nov 15, 2006

Do all of the exercises and problems listed below. Hand in your problem set in the rolling cart hand-in box, either before class or after class, or in the box in the Physics and Astronomy main office by 5 p.m. **Please put your box number on your assignment, which is 952 plus your CPS number**, as well as the course number (Physics 262). Show all your work, write clearly, indicate directions for all vectors, and be sure to include the units! Credit will be awarded for clear explanations as much, if not more so, than numerical answers. Avoid the temptation to simply write down an equation and move symbols around or plug in numbers. Explain what you are doing, draw pictures, and check your results using common sense, limits, and/or dimensional analysis.

10.1 Quantum mechanics: it's where the action is

The constant $\hbar$ introduced by Planck has the units of what is called *action* in physics. In the SI system, action is measured in $J \cdot s$. In terms of physical quantities, it has the units of energy times time, or momentum times distance, or simply angular momentum. In Nature, quantum effects become significant when the typical action scale of the system is on the order of $\hbar$. (In principle, one can engineer quantum systems on any scale, but quantum effects dominate "natural" systems on this scale.) In this problem, we'll explore how $\hbar$ compares to the typical action in some physical systems.

(a) How much action is associated with a baseball thrown by a typical major league pitcher to a catcher at home plate? State any assumptions or approximations you make. (Hint: Consider various physical quantities associated with the situation and combine them to get something having units of action. For example, some quantities that might be useful are the baseball's energy, its momentum, its time of flight, its mass, the distance to home plate, etc., etc.)

(b) How much action is associated with a bicycle tire for a typical Tour de France cyclist? As in part (a), state any assumptions or approximations you make.

(c) How much action is associated with the Moon orbiting the Earth? As in part (a), state any assumptions or approximations you make.

(d) Express your answers in parts (a), (b), and (c) as multiples of $\hbar$. This should make it clear why we don't use quantum mechanics to study these things (usually).

10.2 All things great and small

(a) Three fundamental constants of nature, Planck's constant $\hbar$, Newton's gravitational constant G and the speed of light c can be combined to form a quantity having units of mass, a quantity having units of length, and a quantity having units of time. These are called the Planck mass m_P, the Planck length ℓ_P, and the Planck time t_P respectively. Our current theory of physics breaks down for masses greater than m_P, lengths smaller than ℓ_P, and times shorter than t_P. Use dimensional analysis to derive expressions for m_P, ℓ_P, and t_P in terms of $\hbar$, G, and c. Show your work.

(b) When we discussed special relativity, we saw that the constancy of the speed of light in all inertial reference frames allows one to reasonably use c as a conversion factor between units of time and units of space. One could use this, for example, to assert that so many meters of time had passed since one's last birthday. In a similar fashion, one can combine $\hbar$, G, and c together to form conversion factors allowing one to measure virtually any physical quantity in meters, seconds, or kilograms—whichever unit one chooses. For each of the quantities below, express them first solely in meters, then solely in seconds, then solely in kilograms.

(i) The mass of the Sun.
(ii) The diameter of the moon.
(iii) A year.
(iv) The mass of an electron.

10.3 The surface temperature of the Sun

Planck's blackbody radiation law proved to be an exceptional boon to the field of astrophysics, as it enabled various properties of stars that were obviously too far away to measure directly to be inferred indirectly. In this problem, we show how this law can be used to determine the surface temperature of the Sun by using only terrestrial measurements.

As you (hopefully!) know by now, the Earth revolves around the Sun with the period of a year and rotates about its own axis with the period of a day. At any given moment, the Earth presents only a hemisphere's worth of its surface to the Sun. (The "day side.") Because the Earth is so far from the Sun, the light rays it receives are essentially all parallel to one another. Using this approximation, the total solar power P_{SE} that the Sun imparts to the Earth can be calculated via the following flux integral of the Sun's intensity over the Earth's day side, similar to the flux integrals we considered in electromagnetism:

$$\begin{aligned}
P_{SE} &= \int \vec{\mathbf{I}}_{SE} \cdot \hat{\mathbf{n}}\, da \\
&= I_{SE} \int -\hat{\mathbf{z}} \cdot \hat{\mathbf{r}}\, da \\
&= I_{SE} \int (0,0,-1) \cdot (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\, r^2 \sin\theta d\theta d\phi|_{r=R_E} \\
&= I_{SE} \int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi/2} -R_E^2 \cos\theta \sin\theta d\theta d\phi \\
&= -I_{SE} A,
\end{aligned}$$

where $\vec{\mathbf{I}}_{SE}$ is the (vector) intensity flux of the Sun's light striking the Earth, I_{SE} its magnitude, R_E is the radius of the Earth, and A is the effective surface area the Earth presents to the Sun. The minus sign indicates that the flux of power from the Sun is going *into* the hemisphere of the Earth facing the Sun. With this convention understood, it is the magnitude of the flux that is the relevant quantity.

(a) Calculate A by performing the integral over θ and ϕ above.

(b) What is the total radiant power P_S emitted by the Sun in terms of variables representing the radius of the Sun R_S, its surface temperature T_S, and the Stefan-Boltzmann constant σ_B? Take the emissivity of the Sun to be 1.

(c) What is the intensity I_{SE} of this light at a distance d from the Sun in terms of the variables R_S, T_S, d, and the constant σ_B? (Here d represents the mean distance the Earth has from the Sun in its orbit. Note that because $d \gg R_E$, the difference between measuring d from the center of the Sun versus its surface is negligible, so just consider it to be measured from the center of the Sun for simplicity.)

(d) The radiant power P_{SE} the Earth receives from the Sun is partially reflected and partially absorbed. The portion absorbed is $P_{SE}^{abs} = (1-a)P_{SE}$, where a is the *albedo* of the Earth's surface. Using the expressions for P_{SE}, I_{SE}, and A above, express P_{SE}^{abs} in terms of R_S, T_S, d R_E, a, and σ_B.

(e) Because the Earth is a good blackbody radiator (emissivity $\epsilon \approx 1$) that is in thermal equilibrium with the radiation it receives from the Sun, it reradiates all the solar power it absorbs on its day side over its entire surface (both day and night sides of the Earth). What is the total radiant power P_E emitted by the Earth in terms of R_E, σ_B, and its surface temperature T_E?

(f) As mentioned in the previous part, in thermal equilibrium the Earth emits all of the radiant power it absorbs. Equate P_E and P_{SE}^{abs} from parts (d) and (e) and solve for T_S as a function of T_E and other variables. As a check that your work on the previous parts is correct, the answer you should find is

$$T_S = (1-a)^{-1/4}\sqrt{\frac{2d}{R_S}}\,T_E.$$

(g) For the Earth and Sun today, $a \approx 0.35$, and $T_E \approx 287$ K. Use the table in the back flap of your textbook for astronomical distances and combine them with these values to compute the surface temperature of the Sun using the blackbody model we have developed in this problem. What is the fractional discrepancy $f = (T_{\text{model}} - T_{\text{actual}})/T_{\text{actual}}$ that this temperature has with the actual ($i.e.$widely accepted) surface temperature of the Sun of $T_{\text{actual}} = 5780$ K?

(h) One reason for the discrepancy between what this blackbody model predicts for the surface temperature of the Sun and what it is known to be using better methods is that this model does not account for the *greenhouse effect* whereby atmospheric gases trap some of the radiation that the Earth emits that otherwise would have emitted into outer space. Using the more accepted value of $T_S = 5780$ K for the surface temperature of the Sun, what does our blackbody model predict for the surface temperature of the Earth? How much colder is this than the actual surface temperature of the Earth, in degrees Fahrenheit?

(i) During one of the Earth's previous ice ages, the increased coverage of the surface of the Earth by ice and snow increased its albedo dramatically. Using the albedo of snow, which is $a \approx 0.8$ and assuming the entire surface of the Earth was covered with snow during such an ice age, how much colder does our blackbody model predict the Earth was during an ice age relative to today? (Assume the Sun's surface temperature was the same then as it is now.) Express this temperature decrease in degrees Fahrenheit to get a sense of what this would feel like. Brrr!

10.4 The temperature of the universe

The universe is filled with thermal radiation, emitted at the "time of last scattering," approximately 380,000 years after the big bang. The peak wavelength of this radiation is about 1 mm.

(a) What is the effective temperature of this radiation?

(b) What is the energy (in eV) of light quanta at the peak wavelength?

(c) In what region of the electromagnetic spectrum does this peak wavelength occur? (See Fig. 32.20 in your textbook.)

10.5 Humans: Food in, radiation out.

The average human has a surface area of about 2 m^2 and an emissivity of $\epsilon \approx 1$. The average skin temperature of a human is about 90° F, but clothing reduces the surface temperature by about 5° F.

(a) In a comfortable environment of 72° F, at what rate does the average clothed human radiate energy, in watts?

(b) How many (food) calories are radiated by the human in part (a) in one day? (1 food calorie = 1 kcal.) How does this compare to the daily (food) calorie intake assumed in the "Nutrition Facts" label on packaged foods sold in the US?

(c) What is the peak wavelength of light emitted by the human in part (a)? In what part of the electromagnetic spectrum does this fall? (See Fig. 32.20 in your textbook.)

Extra Credit Problems

10.6 A dim bulb

Imagine you are standing 1 m away from a 60 W incandescent lightbulb. For such a bulb, 3 W of the light emitted falls within the visible part of the spectrum while 57 W does not, most of this remainder falling within the infrared part of the spectrum. The average wavelength of the visible light the bulb emits is 550 nm.

(a) How many visible photons are produced each second by the lightbulb?

(b) Modeling the bulb as a sphere radiating uniformly in all directions and modeling the pupil of your eye as a disc of diameter 1 mm, how many photons per second enter your eye?

(c) While the retina of your eye can respond to a single visible light photon, neural filters only allow a signal to pass to the brain to trigger a conscious response when at least about 5 photons arrive within 100 ms. (This is a good thing—we would see too much visual "noise" in low light otherwise.) How far would you have to stand from this 60 W bulb so that you were just at the threshold of being able to see light from it?

10.7 Compton effect: don't forget the electron!

As discussed in class, the Compton effect refers to a reduction in the frequency of (X-ray) light that is scattered off of a stationary electron (in the lab frame).

(*a*) The magnitude of the frequency reduction depends on θ, the angle the scattered light's 3-momentum forms with the the light's initial 3-momentum vector. For the total 4-momentum in this collision to be conserved, the electron must also recoil with a 3-momentum that forms some angle ϕ with the light's initial 3-momentum. Show that ϕ and θ are related by

$$\cot\frac{\theta}{2} = \left(1 + \frac{\hbar\omega}{m_e c^2}\right)\tan\phi$$

(*b*) Express the energy of the scattered electron as a function of $\hbar$, ω, m_e, c, and θ.

(*c*) The Compton effect is the dominant interaction mechanism in living tissue externally bombarded with high-energy light rays (such as X-rays or gamma rays). The energy transfer of light to electrons in tissue is not significant until the photon energies are in excess of about 100 keV. To see this, calculate the maximal percentage of initial photon energy that can be transferred to an electron in an elastic collision (using your formula from part (*b*) when (*i*) the photon's energy is 5.11 keV, and (*ii*) the photon's energy is 5.11 MeV.