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(12.1) (a) For a particle of charge e, mass m that is accelerated by a potential V, conservation of energy demands

$$E_{final} = E_{init} \Rightarrow K.E. = eV$$

In special relativity, KE and E are related by

$$E = KE + mc^2 \Rightarrow E = eV + mc^2$$

Energy and momentum are related in SR as well:

$$E^2 = p^2c^2 + m^2c^4 \Rightarrow p = \pm\sqrt{E^2/c^2 - m^2c^2} =$$

$$\Rightarrow |p| = \sqrt{\frac{(eV+mc^2)^2}{c^2} - m^2c^2} = \sqrt{\frac{(eV)^2}{c^2} + 2meV} = \sqrt{2meV}\left(1 + \frac{eV}{2mc^2}\right)^{1/2}$$

By the de Broglie relation, the wavelength of the particle is

$$\boxed{\lambda = \frac{h}{|p|} = \frac{h}{\sqrt{2meV}}\left(1 + \frac{eV}{2mc^2}\right)^{-1/2}}$$

(b) In the nonrelativistic limit, KE and p are related by

$$KE = \frac{p^2}{2m} \Rightarrow p = \sqrt{2m \cdot KE} = \sqrt{2meV}$$

So the nonrelativistic de Broglie wavelength is $\lambda_{NR} = \frac{h}{\sqrt{2meV}}$.

Using the binomial expansion $(1+x)^n \approx 1 + nx$ for $x \ll 1$, for $eV \ll 2mc^2$ (the nonrelativistic limit),

$$\left(1 + \frac{eV}{2mc^2}\right)^{-1/2} \approx 1 - \frac{eV}{4mc^2} \approx 1.$$

So $\boxed{\lambda_{SR} = \frac{h}{\sqrt{2meV}}\left(1 + \frac{eV}{2mc^2}\right)^{-1/2} \approx \frac{h}{\sqrt{2meV}} \approx \lambda_{NR}}$

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12.1 (c) For the relativistic an nonrelativistic expression to be in error by more than f, we must have

$$f \leq \frac{|\lambda_{NR} - \lambda_{SR}|}{\lambda_{SR}} = \frac{\lambda_{NR} - \lambda_{SR}}{\lambda_{SR}} \qquad \left(\lambda_{SR} < \lambda_{NR} \text{ b/c } \frac{h}{\gamma m v} < \frac{h}{mv} \right)$$

Using the expressions from parts (a) and (b) we have

$$f \leq \frac{\cancel{\frac{h}{\sqrt{2meV}}}\left(1 - \left(1 + \frac{eV}{2mc^2}\right)^{-1/2}\right)}{\cancel{\frac{h}{\sqrt{2meV}}}\left(1 + \frac{eV}{2mc^2}\right)^{-1/2}}$$

$$\left(1 + \frac{eV}{2mc^2}\right)^{-1/2}(f+1) \leq 1$$

$$(1+f)^2 \leq 1 + \frac{eV}{2mc^2} \qquad (\text{using } f<1)$$

$$eV \geq 2mc^2\left((f+1)^2 - 1\right) = 2mc^2(f^2+2f)$$

(i) For an electron, $mc^2 = 0.511$ MeV. Using $f = 1\% = 0.01$:

$$E = eV \geq 2mc^2(f^2+2f) = 2(0.511\text{ MeV})(2\times10^{-2} + 10^{-4})$$

$$E \geq 2.05\times10^{-2}\text{ MeV} = \boxed{20.5\text{ keV}}$$

(ii) For a neutron (which can't be accelerated by an electric potential, but could acquire the associated energy by some other means), $mc^2 \approx 939.6$ MeV. Using $f = 0.01$ again:

$$E \geq 2(939.6\text{ MeV})(2\times10^{-2} + 10^{-4}) = \boxed{37.7\text{ MeV}}$$

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12.2 (a) From problem 12.1, at very high energies the relativistic expression for the de Broglie wavelength must be used. For eletrons, the cutoff wavelength below which the relativistic formula must be used to not be in error by more than 1% is

$$E = \frac{hc}{\lambda} \geq E_{crit} \Rightarrow \lambda \leq \frac{hc}{E_{crit}} = \frac{1240\,eV\cdot nm}{20.3 \times 10^3\,eV} = 6.1 \times 10^{-2}\,nm$$

$$\lambda \leq 61\ pm$$

So for atoms and viruses I will use the nonrelativistic formula, but for the proton I will use the fully relativistic formula,

To image an object of linear dimension d, we must use a probe whose de Broglie wavelength is $\lambda \leq d$. For electrons this yields the formula:

Non relativistic:

$$d \geq \lambda = \frac{h}{\sqrt{2meV}} \Rightarrow V \geq \frac{h^2}{2med^2}$$

(i) Virus:

$$V \geq \frac{(hc)^2}{2mc^2 ed^2} = \frac{(1240\,eV\cdot nm)^2}{2(0.51 \times 10^6\,eV)\,e\,(10\,nm)^2}$$

$$V \geq 1.5 \times 10^{-2}\,V = \boxed{15\,mV}$$

(ii) Atom:

$$V \geq \frac{(1240\,eV\cdot nm)^2}{2(0.511 \times 10^6\,eV)\,e\,(0.1\,nm)^2} = \boxed{150\ V}$$

A nonrelativistic calculation of V for the proton would say we need 1.5 TV, but we know this will be wrong at the 1% level. (Actually, as we shall see, it is wrong at the 10,000% level)

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12.2 (a) Relativistic calculation:

$$d \geq \lambda = \frac{h}{\sqrt{2meV}}\left(1 + \frac{eV}{2mc^2}\right)^{-1/2}$$

$$1 + \frac{eV}{2mc^2} \geq \frac{h^2}{2meVd^2}$$

$$\left(\frac{e}{2mc^2}\right)V^2 + V - \frac{h^2}{2med^2} \geq 0$$

$$V \geq \frac{mc^2}{e}\left(-1 \pm \sqrt{1 + \frac{2e}{mc^2}\frac{h^2}{2med^2}}\right) = \frac{mc^2}{e}\left(-1 \pm \sqrt{1 + \frac{h^2}{m^2c^2d^2}}\right)$$

Check: for $h \ll m^2c^2d^2$, this reduces to

$$V \geq \frac{mc^2}{e}\left(-1 \pm \left(1 + \frac{1}{2}\frac{h^2}{m^2c^2d^2}\right)\right) \geq$$

$$\geq \frac{\cancel{mc^2}}{e}\frac{h^2}{2m^{\cancel{2}}\cancel{c^2}d^2} = \frac{h^2}{2med^2}$$, the nonrelativistic formula.

So we have:

(iii) Proton: $V \geq \frac{(0.511 \times 10^6\, eV)}{e}\left(-1 + \sqrt{1 + \frac{(1240\, eV \cdot nm)^2}{(0.511 \times 10^6\, eV)^2 \left(10^{-15}\, m \times \frac{10^9\, nm}{1\, m}\right)^2}}\right)$

$$\boxed{V \geq 1.24\ GV}$$

Notice how radically different this is from the 1.5 TV that would be predicted by the nonrelativistic formula — nearly a factor of 1000 times smaller energy is required!

(In the next part, we will see that the energy required by photons to image protons is nearly the same as for such ultrarelativistic electrons.)

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12.2 (b) For photons, $E = pc$, so the de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{E} \iff E = \frac{hc}{\lambda}.$$

Again setting $\lambda \sim d$ for resolvability we have

(i) Virus: $E = \dfrac{1240\text{ eV}\cdot\text{nm}}{10\text{ nm}} = \boxed{124\text{ eV}}$

(ii) Atom: $E = \dfrac{1240\text{ eV}\cdot\text{nm}}{0.1\text{ nm}} = \boxed{12.4\text{ keV}}$

(iii) Proton: $E = \dfrac{1240\text{ eV}\cdot\cancel{\text{nm}}}{1\,\cancel{\text{fm}} \left|\dfrac{\cancel{\text{m}}}{10^{15}\,\cancel{\text{fm}}}\right|\dfrac{10^{9}\,\cancel{\text{nm}}}{\cancel{\text{m}}}} = \boxed{1.24\text{ GeV}}$

(c) Considering the wavelengths needed for resolvability, for a virus UV light is needed, for an atom X-rays are needed, and for a proton γ-rays are needed. This last case is very impractical, so an electron microscope would be better. (Of course, the electron microscope is therefore a reasonable alternative for the first two cases as well.)

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12.3 (a) Using $E = mc^2$ and $\Delta E \Delta t \geq \hbar/2$, we have

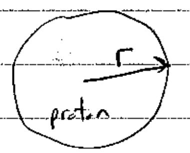
$$\Delta m \geq \frac{\hbar}{2\Delta t c^2}.$$

For $\Delta t = 7.6 \times 10^{-21}$ s, $m = 3.097$ GeV/c^2, the fractional mass uncertainty is

$$\frac{\Delta m}{m} = \frac{\hbar}{2mc^2\Delta t} = \frac{h}{8\pi mc^2 \Delta t} = \frac{6.626\times10^{-34}\,\text{J s}}{4\pi(3.097\times10^{9}\,\text{eV})(7.6\times10^{-21}\,\text{s})}\,\frac{\text{eV}}{1.6\times10^{-19}\,\text{J}}$$

$$= 1.4\times10^{-5} = \boxed{1.4\times10^{-3}\ \%}$$

(b)



if e^- is in a proton, $\Delta x = 2r$

By uncertainty principle, $\Delta x \Delta p \geq \hbar/2$

Approximating $x \sim \Delta x$, $p \sim \Delta p$, a nonrelativistic calculation at this Heisenberg limit yields:

$$KE = \frac{p^2}{2m} = \frac{\hbar^2}{8mx^2} = \frac{\hbar^2}{32mr^2} = \frac{(\hbar c)^2}{32mc^2r^2} = \frac{(1240\,\text{eV}\cdot\text{nm})^2}{32(0.511\times10^6\,\text{eV})(10\times10^{-15}\,\text{m}\,\frac{10^9\,\text{nm}}{\text{m}})^2}$$

$$KE = 9.4\times10^{8}\,\text{eV} = 940\,\text{MeV}$$

If we used the more accurate relativistic calculation, we would find

$$KE = E - mc^2 = \sqrt{p^2c^2 - m^2c^4} - mc^2 = mc^2\left(\sqrt{\frac{p^2}{m^2c^2} - 1} - 1\right)$$

$$= mc^2\left(\sqrt{\frac{\hbar^2c^2}{4x^2m^2c^4} - 1} - 1\right) = mc^2\left(\sqrt{\frac{(\hbar c)^2}{16r^2(mc^2)^2} - 1} - 1\right)$$

$$= (0.511\times10^6\,\text{eV})\left(\sqrt{\frac{(1240\,\text{eV}\cdot\text{nm})^2}{16(10^{-5}\,\text{nm})^2(0.511\times10^6\,\text{eV})^2} - 1} - 1\right)$$

(i) $\boxed{KE = 30\ \text{MeV}}$

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12.3 (b) (ii) Compared to the neutron's rest mass of 939.6 MeV, this is

$$\frac{KE}{m_n c^2} = \frac{30 \times 10^6 \, eV}{939.6 \times 10^6 \, eV} = \boxed{3.193 \times 10^{-2}} \cong 3\%$$

Since the difference in proton and neutron rest masses is

$$m_n c^2 - m_p c^2 = (939.56536 - 938.27203) \times 10^6 \, eV$$

$$= 1.293 \, MeV$$

$$\ll E_{electron} = KE + mc^2 \cong 31 \, MeV,$$

it can't explain the mass difference solely in terms of an electron being in a proton. (The electron would escape!)

(c) (i) Using $E = mc^2$ and $\Delta E \Delta t \geq \hbar/2$, we have at the Heisenberg limit

$$\Delta t = \frac{\hbar}{2 \Delta mc^2} = \frac{6.57 \times 10^{-16} \, eV \cdot s}{2(135 \times 10^6 \, eV)} = \boxed{2.4 \times 10^{-24} \, s} \quad \text{(short!)}$$

Travelling at the speed of light for this time covers a distance of

$$\Delta x = c \Delta t = (3 \times 10^8 \, m/s)(2.4 \times 10^{-24} \, s) = 7.3 \times 10^{-16} \, m$$

$$= \boxed{0.73 \, fm}$$

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12.4 Hydrogen atom: momentum p, radius r

(a) $$\boxed{E_{TOT} = \frac{p^2}{2m} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}}$$

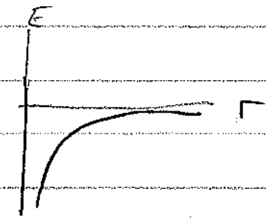
(b) Newton's 2nd Law: $\sum \vec{F} = m\vec{a}$

Here: $$-\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\hat{r} = -\frac{mv^2}{r}\hat{r} \iff \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} = \frac{p^2}{m}$$

$$\Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}\left(\frac{1}{2} - 1\right) =$$

$$\boxed{E = -\frac{1}{8\pi\varepsilon_0}\frac{e^2}{r}}$$

Min energy: $\boxed{r = 0} \Rightarrow E = -\infty$



(c) Using $\Delta p \Delta x \geq \hbar/2$ and $\Delta p \sim p$, $\Delta x \sim x = 2\pi r$, we have

$$2\pi p r = \hbar/2 \Rightarrow p = \frac{\hbar}{4\pi r}$$

$$\Rightarrow E_{min} = \left(\frac{\hbar}{4\pi r}\right)^2 \frac{1}{2m} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}$$

$$\boxed{E_{min} = \left(\frac{\hbar^2}{32\pi^2 m}\right)\frac{1}{r^2} - \left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{r}}$$

(d) $$\frac{dE_{min}}{dr} = 0 = -2\left(\frac{\hbar^2}{32\pi^2 m}\right)\frac{1}{r^3} + \left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{\cancel{r^2}}$$

$$r = \frac{\hbar^2}{4\cdot 16\pi^2 m}\frac{\cancel{4\pi}\varepsilon_0}{e^2} = \boxed{\frac{\hbar^2\varepsilon_0}{4\pi m e^2} = r}$$

The Bohr radius for H is $a_0 = \frac{h^2\varepsilon_0}{\pi m e^2} = \frac{4\pi^{\cancel{2}}\hbar^2\varepsilon_0}{\cancel{\pi} m e^2} = \frac{4\pi\hbar^2\varepsilon_0}{m e^2}$

Hence $$\boxed{r = \frac{a_0}{16\pi^2}}$$

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12.5 (a) To normalize $\Psi(x,t) = A e^{-a[(mx^2/\hbar)+it]}$

$$1 = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx$$

$$= A^2 \int_{-\infty}^{\infty} e^{-2amx^2/\hbar} \, dx \qquad \alpha = \frac{2am}{\hbar}$$

$$= A^2 \sqrt{\frac{\pi\hbar}{2am}}$$

$$\Rightarrow \boxed{A = \left(\frac{2am}{\pi\hbar}\right)^{1/4}}$$

(b) The Schrödinger equation applied to this wave function is

$$i\hbar \frac{\partial \Psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + V\Psi$$

$$i\hbar(-ia)\Psi = \frac{-\hbar^{\cancel{2}}}{\cancel{2m}} \frac{\partial}{\partial x}\left(-\frac{\cancel{2am}x}{\cancel{\hbar}}\Psi\right) + V\Psi$$

$$\cancel{\hbar a \Psi} = \hbar a\left(\cancel{\Psi} + x\left(\frac{-2max}{\hbar}\right)\cancel{\Psi}\right) + V\cancel{\Psi}$$

$$\boxed{V = 2ma^2x^2}$$ (Note that the full Schrödinger equation must be used here.)

(c) (i) $\langle x \rangle = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x \, dx = 0$ because the integrand is an odd fn.

(ii) $\langle x^2 \rangle = \int_{-\infty}^{\infty} |\Psi(x,t)|^2 x^2 \, dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-2amx^2/\hbar} \, dx$

Using $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = -\frac{d}{d\alpha}\left(\pi^{1/2}\alpha^{-1/2}\right) = +\frac{1}{2}\pi^{1/2}\alpha^{-3/2}$, this becomes

$$\langle x^2 \rangle = \frac{1}{2}\left(\frac{\cancel{\pi}\hbar^{\cancel{3}\,2}}{4\cancel{8}a^{\cancel{3}\,2}m^{\cancel{3}\,2}}\right)^{1/2}\left(\frac{\cancel{2am}}{\cancel{\pi\hbar}}\right)^{1/2} = \frac{1}{2}\left(\frac{\hbar^2}{4m^2a^2}\right)^{1/2} = \frac{\hbar}{4ma}$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{4ma}}$$

(iii) $\boxed{\Delta x = \left(\frac{\hbar}{4ma}\right)^{1/2}}$