Phase PS 1 Solutions

Search coils and credit cards

Search coil:

\[ \begin{align*}
\text{Area: } A \\
\text{Turns: } N \\
\text{Direction of area: } \hat{n} \text{ into page} \quad (\hat{n} = -\hat{z})
\end{align*} \]

Initial \( \vec{B} \) field:
\( \vec{B}_i = B \hat{z} \) (out of page)

Final \( \vec{B} \) field:
\( \vec{B}_f = 0 \)

Time for \( \vec{B} \) change:
\( \Delta t \)

(a) Faraday's Law:
\[ E = -\frac{d}{dt} \oint \vec{B} \cdot \hat{n} \, da = -\frac{d\Phi_b}{dt} \]

Applied to this problem:

Initial flux:
\[ \Phi_b^i = \oint \vec{B} \cdot \hat{n} \, da \]
\[ = B \hat{z} \cdot (-\hat{z}) (\text{NA}) = -BNA \]

Final flux:
\[ \Phi_b^f = 0 \]

Emf:
\[ E = -\frac{\Delta \Phi_b}{\Delta t} = \frac{BNA}{\Delta t} \]

Emf direction:
Clockwise (\( \left( \frac{\mathcal{C}}{\mathcal{C}} \right) \))

(By right hand rule around \( \hat{n} \))

Lenz' Law check: Current creates \( \vec{B} \) field to oppose change in external \( \vec{B} \) field — OK
Ohm's Law: $E = IR$

Total charge that flows through coil in time $\Delta t$:

$$dQ = I \, dt$$

$$\int_0^\Delta Q = \int_0^\Delta I \, dt$$

$$Q = I \Delta t$$

Relating $Q$ to emf $E$:

$$E = \frac{d\Phi}{dt} = I \, R = \frac{dQ}{dt}$$

(a) $Q = \frac{BNA}{R}$

(b) A credit card reader is a search coil. However, instead of it moving, the card moves through it. It measures the $B$ field, the magnitude of which, by part (a), is proportional to the charge on the card. The information is stored in the charge.

(c) The data is stored in the charge $Q$, so it is independent of the time of the swipe.
The switch $S$ is opened after being closed.

**Analysis:**

The current $I$ in the circuit initially flows as shown in the figure above. When $S$ is opened, the current decreases to zero, so $dI/dt$ goes counterclockwise, opposite to the original direction of $I$.

The helicity of the coils in the left circuit is such that a counterclockwise current in the circuit generates a $B$ field in the solenoid pointing to the left. Hence, a changing current in this direction creates a changing $B$ field in this direction. (See figure)

The changing $B$ field in the left circuit extends beyond that circuit into the region of the right circuit. This creates a changing flux through the coils of the right circuit, such that the flux going to the left through this area is increasing. (At $d\Phi/dt \cdot H > 0$ in the figure)

**Lenz' Law:** The changing flux in the right circuit creates an emf $\mathcal{E}$ around $A$ (by the right hand rule, counterclockwise when viewed from the left) that opposes the time change of this flux:

$$\mathcal{E} = -\frac{d\Phi}{dt} = -A \frac{dB}{dt} \text{ counterclockwise}$$

$$= A \frac{dB}{dt} \text{ clockwise.}$$
By Ohm's Law, the emf $E$ creates a current $I$ in the same direction by $E=IR$. Given the helicity of the coils in the right circuit, this current circulates counterclockwise in the right circuit as viewed from above.

Therefore, the induced current flows from $a$ to $b$.

(b) Coil B is brought closer to coil A with the switch closed.

By the helicity of the winding in coil A, the $\vec{B}$ field from coil A points to the right. As coil B is brought closer to coil A, it experiences an increasing $\vec{B}$ field to the right, and correspondingly an increasing magnetic flux to the right.

\[
\vec{B} \rightarrow \bigoplus \rightarrow \hat{n} \quad A \frac{dB}{dt} \cdot \hat{n} > 0
\]

Coil B

By Faraday's Law, an emf $E$ around $\hat{n}$ (by the right-hand rule, clockwise when viewed from the left) is created that opposes the time change of this flux.

\[
E = -\frac{dB}{dt} = -A \frac{dB}{dt} \text{ clockwise}
\]

= $A \frac{dB}{dt}$ \text{ counter clockwise}
By Ohm's Law, the emf creates a current $I$ in the
same direction by $E = I R$. Given the helicity of coil B,
this current circulates clockwise in circuit B as viewed
from above.

Therefore, the induced current flows from $b$ to $a$.

(c) The resistance $R$ in circuit A is decreased while the switch
remains closed.

As $R$ is decreased, more current flows through circuit
A by Ohm's Law, $E = I R$. This increased current increases
the magnetic field in coil B, which points to the right.
This increases the magnetic flux through coil B to the right.

$$\oint_{\partial A} \mathbf{A} \cdot d\mathbf{l} = \int_{S} A dB \cdot \hat{n} > 0$$

This is the same situation as in part (b), so the same
analysis applies, and therefore

the induced current flows from $b$ to $a$. 

Find the emf in the rod.

Method 1: "Motional emf" (I dislike this method)

\[ |E| = |VBL| = (7.50 \text{ m/s})(0.800 \text{ T})(50.00 \text{ cm}^2 / 100 \text{ cm}) \]

\[ |E| = 3.00 \text{ V} \]

Notice that Method 1 doesn't help you figure out which direction the emf is in. That's a big part of why this method is lousy. A better way to do part (b) that will also answer the direction question here, is the following.

Method 2:

Faraday's Law: 

\[ \mathbf{E} = -\frac{d}{dt} \mathbf{B} \cdot \mathbf{n} \, d\mathbf{a} \]

\[ \mathbf{n} : \text{ into page } (-\hat{x}) \]

\[ \mathbf{B} \cdot \mathbf{n} = \mathbf{B} (-\hat{x}) \cdot (-\hat{x}) = \mathbf{B} \]

\[ \mathbf{B} = \mathbf{S} \mathbf{B} \cdot \mathbf{n} \, d\mathbf{a} = \mathbf{B} \mathbf{L} \mathbf{r} \]
\[ E = -\frac{d\Phi}{dt} = -BL \frac{dt}{dt} = -BLV \]

Direction: right hand rule around \( \hat{z} \) is clockwise

\( \Rightarrow \ E = -BLV \text{ clockwise} \)

\( = BLV \text{ counterclockwise} \)

If the wire is ohmic, \( E = IR \), so

\[ I = \frac{BLV}{R} \text{ counterclockwise} \]

(only the direction was asked for, but faraday's law gives so much more!)

\( \bigcirc \) If \( R = 1.50 \, \Omega \) (assumed constant) find \( F \) needed to keep rod moving at \( V \)

The Lorentz force on the rod is

\[ \vec{F} = I \vec{L} \times \vec{B} \text{, where } \vec{L} = L\hat{z} \text{, in the direction of the curve} \]

\[ \vec{F} = \left( BLV \right) \frac{R}{R} \hat{z} \times B \hat{z} = \left( -\hat{x} \right) \]

\[ = \frac{B^2L^2V}{R} \hat{z} \times \hat{z} = \hat{y} \]

\[ \vec{F} = \frac{B^2L^2V}{R} \hat{y} \]
P26a P51

YF Exercise

Pugging in numbers,

\[ F = \left(0.800 T\right)^2 \left(\frac{50.00 \text{ cm}}{100 \text{ cm}}\right)^2 \left(7.5 \text{ m/s}\right) \hat{y} \]

\[ F = 0.800 N \hat{y} \]

\( d \) Rate of mechanical work: from part c

\[ P_{\text{mech}} = \frac{dW}{dt} = F \cdot v = \left(\frac{B^2 L^2 v}{R}\right) \cdot (v \hat{y}) \]

\[ P_{\text{mech}} = \frac{B^2 L^2 v^2}{R} \]

Pugging in numbers:

\[ P_{\text{mech}} = (0.800 N)(7.50 \text{ m/s}) \]

\[ P_{\text{mech}} = 6.00 \text{ W} \]

Rate of ohmic heating from part b

\[ P_{\text{ohmic}} = J^2 R = \left(\frac{BLv}{R}\right)^2 R \]

\[ P_{\text{ohmic}} = \frac{B^2 L^2 v^2}{R} \]

Algebraically, \( P_{\text{mech}} = P_{\text{ohmic}} \)

Hence, \( P_{\text{ohmic}} = 6.00 \text{ W} \)
Self-emf

Wire Loop: radius $a$, resistance $R$

$\vec{B}_{\text{init}} = -B_{\text{ext}} \hat{z}$

$\vec{B}_{\text{final}} = 0$

$
\hat{n} \text{  (direction of loop area)} = -\hat{z}
$

(1) When the external $\vec{B}$ drops to zero, it creates an emf in the loop by Faraday's Law:

$$E_{\text{loop}} = -\frac{1}{\text{area of loop}} \int \vec{B}_{\text{ext}^+} \cdot \hat{n} \, dA$$

$$= -\left( \frac{\vec{B}_{\text{ext}^+} \cdot \hat{n} - \vec{B}_{\text{ext}^+} \cdot \hat{n}}{\partial t} \right) \times a^2$$

$$= -\left( \frac{0 - (-B_{\text{init}}^2 \times (-\hat{z}))}{\partial t} \right) \times a^2$$

$$= \left( B_{\text{init}}^2 \times \frac{\pi a^2}{\partial t} \right) \text{ clockwise (right hand rule around } \hat{n})$$

For an ohmic wire, the current goes in the same direction as $E$.

Approximating the field from this wire to be uniform in the loop, we find

$$\vec{B}_{\text{loop}} = \frac{\mu_0 i}{\pi a^2} \hat{z}$$
The flux of this field through the loop is

\[ \Phi_{\text{loop}} = \oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{a} \]

\[ = - \frac{\mu_0}{2\pi} \frac{\Delta}{2} (-2) \pi \alpha^2 \]

\[ \Phi_{\text{loop}} = \frac{\pi \alpha^2}{2} \text{ into the page} \] (in -z direction)

(b) By Faraday's Law,

\[ \mathcal{E}_{\text{loop}} = -\frac{d \Phi_{\text{loop}}}{dt} \]

and by Ohm's Law, \( \mathcal{E}_{\text{loop}} = I R \).

Putting these together,

\[ I R = -\frac{d}{dt} \left( \frac{\pi \alpha^2 I}{2} \right) \]

\[ = -\frac{\pi \alpha^2}{2} \frac{dI}{dt} \]

\[ \Rightarrow \frac{dI}{dt} = \left( \frac{2 R}{\pi \mu_0 \alpha^2} \right) I \]
If \( i(t=0) = i_0 \), then we can solve this differential equation:

\[
\frac{d^2 i}{dt^2} = -\left(\frac{2R}{\pi \mu_0 a}\right) i \\
\frac{d^2 i}{i} = -\left(\frac{2R}{\pi \mu_0 a}\right) dt \\
\int_{t=0}^{t} \frac{d^2 i}{i} = -\left(\frac{2R}{\pi \mu_0 a}\right) \int_{0}^{t} dt
\]

\[
\ln\left(\frac{i}{i_0}\right) = -\left(\frac{2R}{\pi \mu_0 a}\right) t
\]

Exponentiate both sides, use \( e^{\ln x} = x \):

\[
\frac{i}{i_0} = e^{-\left(\frac{2R}{\pi \mu_0 a}\right) t}
\]

\[
i = i_0 e^{-\left(\frac{2R}{\pi \mu_0 a}\right) t}
\]

(a) For \( a = 50 \text{ cm} \), \( R = 0.10 \Omega \), when is \( i = 0.010 i_0 \)?

To solve this, we take the natural log of the solution in part (6), or more simply just use this formula:

\[
t = \left(\frac{\pi \mu_0 a}{2R}\right) \ln\left(\frac{0.010i_0}{i_0}\right)
\]

\[
t = \left(\pi \left(4\pi \times 10^{-7} \text{ wb/A.m}\right) \left(\frac{50 \text{ cm}}{100 \text{ cm}}\right)^2\right) \ln (0.010)
\]

\[
t = 4.55 \times 10^{-5} \text{ s}
\]

We ignore this effect because we are interested in physics on a much slower time scale than 50 ms.
(a) Find $E_{\text{loop}}$ using (i) Faraday's Law, (ii) Motional emf.

To begin, the current in the wire creates a $\mathbf{B}$ field circulating around it, whose magnitude drops off like $\frac{1}{r}$. In the plane of the loop, I have indicated this by using decreasingly large X's. To find this field, we use Ampere's Law:

$$\oint_{\text{loop}} \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{in}}$$

$$\int_{r=0}^{\infty} B_{\phi}(r) \cdot r dr \hat{\phi} = \mu_0 I$$

$$r B_{\phi}(r) \int_{0}^{\pi} \hat{\phi} \frac{d\theta}{\theta} = \mu_0 I$$

$$2\pi r B_{\phi}(r) = \mu_0 I$$

$$B_{\phi}(r) = \frac{\mu_0 I}{2\pi r}$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi r} \hat{\phi}$$  \hspace{1cm} \text{(Circulates ccw, viewed from above)}
To use Faraday's law, first we calculate the flux through the loop, then we calculate how it changes when it is moving with velocity \( \mathbf{v} \).

Because \( \mathbf{\hat{B}} \) is not constant through the loop, we divide the area into infinitesimally thin strips of width \( \, \mathrm{d}a \) — so thin in fact that \( \mathbf{\hat{B}} \) is constant in each strip. We then add the flux through each strip to find the total flux:

\[
\Phi = \oint \mathbf{\hat{B}} \cdot \mathbf{n} \, \mathrm{d}a
\]

\[
\begin{align*}
\, \mathrm{d}\Phi &= \mathbf{\hat{B}}_{\text{wire}} \cdot \mathbf{n} \, \mathrm{d}a \\
&= \left( \frac{\mu_0 I}{2\pi r} \right) \cdot (-\hat{x}) \left( b \, \mathrm{d}r \right) \\
&= \frac{\mu_0 I b}{2\pi} \, \mathrm{d}r
\end{align*}
\]

\[
\Phi = \left[ \frac{\mu_0 I b}{2\pi} \ln \left( \frac{r_{\text{out}}}{r_1} \right) \right]_{r_1}^{r_{\text{out}}}
\]

\[
\Phi = \left( \frac{\mu_0 I b}{2\pi} \right) \ln \left( \frac{r_{\text{out}}}{r_1} \right) \text{ into page}
\]
The emf in the loop is then calculated by Faraday's law, with a judicious application of the chain rule for taking derivatives:

\[ E = -\frac{d\Phi}{dt} = -\frac{d\Phi}{dr} \frac{dr}{d\tau} \]

\[ = -\left(\frac{\mu_0 I b V}{2 \pi r}\right) \frac{d}{dr} \left( \ln(r+a) - \ln r \right) \frac{dr}{d\tau} \]

\[ = -\left(\frac{\mu_0 I b V}{2 \pi r}\right) \left( \frac{1}{r+a} - \frac{1}{r} \right) \]

\[ = \left(\frac{\mu_0 I b V}{2 \pi a}\right) \left( \frac{r}{r+a} - \frac{1}{r+a} \right) \]

\[ = \left(\frac{\mu_0 I b V}{2 \pi a}\right) \left( \frac{r+a-r}{r+a} \right) \]

\[ \therefore E = \frac{\mu_0 I b V a}{2 \pi r (r+a)} \text{ clockwise} \]

(only magnitude computed, but we can already tell the direction)

(i) Using the motional emf argument, (I dislike this method)

\[ E = \mathbf{\mathbf{\nabla}} \times \mathbf{E} \cdot d\mathbf{\mathbf{l}} \]

\[ \mathbf{\mathbf{\nabla}} \times \mathbf{E} = \mathbf{V} \times \mathbf{B}(r) \cdot (-\mathbf{\mathbf{\hat{z}}}) \]

\[ = \mathbf{V} \mathbf{B}(r) \cdot \mathbf{\hat{z}} \times (-\mathbf{\hat{z}}) \]

\[ = \mathbf{V} \mathbf{B}(r) \cdot \mathbf{\hat{y}} \]

Direction is by right hand rule around \( \mathbf{\mathbf{\hat{z}}} \).
(a)(i) Doing the line integral in four pieces:
\[ \int (v \times B) \cdot dl = \int_B(r_\perp \cdot b(-\gamma)) \, dl \]
\[ + \int_{\gamma} B(r) \cdot r_\perp \cdot \hat{\gamma} \, ds \]
\[ + \int_{\gamma} B(r + a) \cdot \hat{\gamma} \cdot b(\hat{\gamma}) \, ds \]
\[ + \int_{\gamma} B(r) \cdot \hat{\gamma} \cdot (-\hat{\gamma}) \, ds \]
Terms 2 and 4 cancel, leaving
\[ \mathcal{E} = \int_B (B(r + a) - B(r)) \cdot v_b \, dl \]
\[ = \frac{\mu_0 v B}{2 \pi} \left( \frac{r + a}{r} - \frac{r}{r + a} \right) \]
\[ = \frac{\mu_0 v B}{2 \pi} \left( \frac{r - (r + a)}{r(r + a)} \right) \]
\[ \mathcal{E} = -\frac{\mu_0 I B v a}{2\pi r(r + a)} \] counter-clockwise (direction of line integral I chose)
\[ \mathcal{E} = \frac{\mu_0 I B v a}{\pi r(r + a)} \] clockwise (Same as (a)(ii))

(b)(i) We already found the emf direction, and ohmic materials have the current moving in the same direction. The intuition of "Lenz' Law" agrees with this; a clockwise current will create a B field into the page, compensating for the loss of B field through the loop that occurs as the loop moves away from the wire.
Using the Lorentz force formula,

$$\vec{F} = I \vec{L} \times \vec{B},$$

The only force along the direction of the wire occurs in the two vertical parts of the loop (at length $b$). Because the $\vec{B}$ field is larger on the part closer to the wire, the net force is given by the force here, which generates a clockwise emf. See the figure below.

![Diagram showing forces and emf](image)

0) Sanity checks:

i) If $v \to 0$, the emf is

$$E = \lim_{v \to 0} \frac{M_0 I b v a}{2 \pi r (c/\mu)} = 0$$

$\checkmark$ No motion, no emf

ii) If $a \to 0$, the emf is

$$E = \lim_{a \to 0} \frac{M_0 I b v a c}{2 \pi r (c/\mu) (c/\mu)} = 0$$

$\checkmark$ No area, no flux

iii) If $r \to \infty$, the emf is

$$E = \lim_{r \to \infty} \frac{M_0 I b v a}{2 \pi r (c/\mu) (c/\mu)} = 0$$

$\checkmark$ No $\vec{B}$, no flux
Bar: length $L$
Mass $M$
Resistance $R$

Fictionless inclined with rails of negligible resistance.

Coordinate system:

Uniform $\mathbf{B}$ downwards

(a) Bar slides down: Flux downwards through closed circuit decreases, so by Lenz' law current will be generated to make a $\mathbf{B}$ field pointing down, so current flows clockwise around the loop as viewed from above.

Hence, the current goes from $a$ to $b$.

(b) To find the terminal velocity, we use Newton's 2nd Law orthogonally:

\[ \sum F_i = m \cdot \vec{a} \]

Flux due:

\[ \vec{F}_{\text{normal}} + \vec{F}_g + \vec{F}_{\text{emt}} = m \cdot \vec{a} = 0 \text{ at terminal velocity} \]

\[ \vec{F}_{\text{emt}} = I \hat{\mathbf{L}} \times \mathbf{B} \]

\[ = I L \mathbf{\hat{x}} \times \mathbf{B} (\hat{\mathbf{x}}) \]

\[ = -I L B \hat{\mathbf{x}} \]

Ohm's Law: $I = \frac{\varepsilon}{R}$ in $\text{bar}$, so

\[ \vec{F}_{\text{emt}} = -\frac{\varepsilon L B}{R} \hat{\mathbf{x}} \]
We use Faraday's Law to calculate the emf

\[ E = -\frac{d}{dt} \int \mathbf{B} \cdot d\mathbf{A} \]

Must choose a direction for \( \mathbf{\hat{n}} \perp \) to area.

Choose \( \mathbf{\hat{n}} = -\cos \phi \mathbf{\hat{x}} - \sin \phi \mathbf{\hat{z}} \)

\[ \frac{d\mathbf{a}}{dt} = \frac{Ld\mathbf{r}}{dt} = L \mathbf{V} \]

\[ \mathbf{B} \cdot \mathbf{\hat{n}} = -B \mathbf{\hat{z}} \cdot (-\cos \phi \mathbf{\hat{x}} - \sin \phi \mathbf{\hat{z}}) \]

\[ = B \cos \phi \]

\[ E = \int \mathbf{B} \cdot \mathbf{\hat{n}} \frac{d\mathbf{a}}{dt} \]

\[ E = BL \mathbf{V} \cos \phi \text{ clockwise, viewed from above} \]

(agrees w/ Lenz' Law in \( b \))

The Lorentz force is therefore

\[ \mathbf{F}_{\text{Lorentz}} = -B^2 L^2 V \cos \phi \mathbf{\hat{x}} \]

The gravitational force is \( \mathbf{F}_g = -mg \mathbf{\hat{z}} \)
Drawing a free-body diagram:

\[ F_N \cos \phi - mg = 0 \]

\[ F_N = \frac{mg}{\cos \phi} \]

\[ \sum F_y = 0 \quad \Rightarrow \quad F_N \sin \phi - \frac{B^2 L^2 v}{R} \cos \phi = 0 \]

\[ \left( \frac{B^2 L^2}{R} \cos \phi \right) v = \left( \frac{mg}{\cos \phi} \right) \sin \phi \]

\[ v = \frac{mg L}{B^2 L^2} \tan \phi \]

As a vector, \( \vec{v} = v \cos \phi \hat{x} - v \sin \phi \hat{z} \)

(6) The induced current at terminal velocity can be found using the emf formula in (6) and Ohm's law, \( E = i R \):

\[ E = BLV \cos \phi \text{ clockwise from above} \]

\[ I = \frac{E}{R} = \frac{BLV \cos \phi}{R} = \left( \frac{BL \cos \phi}{R} \right) \left( \frac{mg L}{B^2 L^2} \right) \tan \phi \]

\[ I = \frac{mg \tan \phi}{BL} \text{ clockwise as viewed from above} \]
Ph362 PS 1

(a) Electrical energy is converted into thermal energy by Ohmic $I^2R$ heating. At terminal velocity, this rate is

$$ P = I^2R $$

$$ P = \frac{m^2g^2R}{8^2L^2} \tan^2\phi $$

(b) The rate of work being done on the bar by gravity is

$$ \frac{dW_g}{dt} = F_g \cdot \dot{v} $$

$$ = (-mg^2) \cdot (v \cos \phi \cdot x - v \sin \phi \cdot \dot{z}) $$

$$ = mgv \sin \phi $$

$$ = (mg \sin \phi) \left( \frac{m g R}{8^2 L^2} \right) \tan^2 \phi \cos \phi $$

$$ \frac{dW_g}{dt} = \frac{m^2 g^2 R}{8^2 L^2} \tan^2 \phi $$

The same as (d). Gravity is causing the $I^2R$ heating, not the Lorentz force.

(This makes sense because we know that magnetic fields can do no work.)
(1) High frequency capacitors

(2) To first approximation, no \( \vec{B} \) fields exist in the capacitor, and Faraday's Law says:

\[
\oint \vec{E} \cdot d\vec{S} = \frac{d}{dt} \int \vec{B} \cdot d\vec{a} = 0
\]

This means that the electric field is conservative to first approximation, so that the electric field is expressible as (minus) the gradient of the potential:

\[
\vec{E}_1 = -\nabla V
\]

\[
= -\frac{V_{\text{top}} - V_{\text{bottom}}}{\Delta z}
\]

\[
\vec{E}_1 = -\frac{V_0}{a} \sin \alpha \hat{z}
\]
Applying the Ampere-Maxwell law to a loop inside the capacitor and coaxial with it,

\[ \oint \vec{B} \cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt} \int \vec{E} \cdot \hat{n} \, d\alpha \]

\[ 2\pi r \vec{B}_1 = \mu_0 \varepsilon_0 \frac{d}{dt} \frac{V_0}{2a^2} (\sin \omega t) \quad \text{use } \mu_0 \varepsilon_0 = \frac{1}{c^2} \]

\[ \vec{B}_1 = -\frac{\omega}{2a^2} \frac{V_0}{2} \cos \omega t \hat{\phi} \quad \text{(clockwise, viewed from above)} \]

Applying Faraday's law to the loop shown, we can find \( \vec{E}_2 \):

\[ \oint \vec{E}_2 \cdot d\vec{l} = \frac{d}{dt} \int \vec{B} \cdot \hat{n} \, d\alpha \]

Using \( \vec{E} = \vec{E}_1 + \vec{E}_2 \), we see the line integral of \( \vec{E}_1 \) vanishes from part (a). There is no correction at the center, so \( \vec{E}_2 = 0 \) at \( r = 0 \). Moreover, the top and bottom parts of this line integral cancel by symmetry.
Faraday's law then gives:

\[
\oint \mathbf{E} \cdot d\mathbf{s} = \mathbf{E}_2 \cdot \mathbf{\hat{n}}
\]

\[
= \mathbf{E}_2 d
\]

\[
= \frac{d}{dt} \oint \mathbf{B} \cdot \mathbf{\hat{n}} \, da
\]

\[
\oint \mathbf{B} \cdot \mathbf{\hat{n}} \, da = \oint \left( -\frac{\omega}{4\pi} \right) \mathbf{V}_0 \cos \omega t \cdot \mathbf{\hat{z}} \cdot \mathbf{\hat{r}} \, dr
\]

\[
= \left( -\frac{\omega}{4\pi} \right) \mathbf{V}_0 \cos \omega t \oint \mathbf{\hat{z}} \cdot \mathbf{\hat{r}} \, dr
\]

\[
= \left( -\frac{\omega}{4\pi} \right) \mathbf{V}_0 \cos \omega t
\]

Thus,

\[
\mathbf{E}_2 = \left( -\frac{\omega}{4\pi} \right) \left( \frac{\mathbf{V}_0}{d} \right) \sin \omega t \mathbf{\hat{z}}
\]

Applying the Ampère-Maxwell law to the same loop as in part (b),

\[
\oint \mathbf{E} \cdot d\mathbf{l} = \frac{1}{c} \frac{d}{dt} \oint \mathbf{B} \cdot \mathbf{\hat{n}} \, da
\]
(cont.)

\[ 2 \pi r \mathbf{B}_2 = \frac{1}{c} \left( \frac{\partial}{\partial t} \left( \frac{\mathbf{v}_4}{a} \right) \right) \mathbf{S} \mathbf{r} \left( \omega t - dr \right) \frac{1}{a} \sin \omega t \]

\[ \mathbf{B}_2 = \frac{\omega^2 r^3}{16 c^4} \left( \frac{\mathbf{v}}{a} \right) \cos \omega t \]

\[ \mathbf{B}_3 = \frac{\omega^3 r^3}{16 c^4} \left( \frac{\mathbf{v}}{a} \right) \cos \omega t \hat{\mathbf{a}} \quad \text{(clockwise, viewed from above)} \]

By Faraday's law on the loop in \( \Theta \):

\[ \oint \mathbf{E}_3 \cdot d\mathbf{s} = -\frac{\mathbf{d}}{dt} \oint \mathbf{B}_3 \cdot \mathbf{n} \, da \]

\[ -\mathbf{E}_3 \, d\mathbf{s} = \frac{\omega^2}{16 c^4} \frac{\mathbf{v}_0}{a} \left[ \mathbf{S} \mathbf{r} \cdot \mathbf{d} \mathbf{r} \right] \cdot \frac{1}{a} \frac{\mathbf{d}}{dt} \cos \omega t \]

\[ \mathbf{E}_3 = \frac{-\omega^3 r^3}{16 c^4} \frac{\mathbf{v}_0}{a} \frac{1}{a} r^4 \cos \omega t \]

\[ \mathbf{E}_3 = \frac{-\omega^4 r^4}{64 c^4} \frac{\mathbf{v}_0}{a} \frac{1}{a} \sin \omega t \hat{\mathbf{a}} \]

Writing out the series for \( \mathbf{E} \) vs. \( t \) has

\[ \mathbf{E} = -\frac{\mathbf{v}_0}{a} \sin \omega t \left( \frac{1}{\omega^2} \left( \frac{\omega t}{c} \right)^2 + \frac{1}{\omega^4} \left( \frac{\omega t}{c} \right)^4 + \ldots \right) \]

\[ E_1 \quad E_2 \quad E_3 \]

If we worked out the next term, we would find

\[ E_4 = -\frac{1}{2^4 4! 6} \left( \frac{\omega t}{c} \right)^6 \frac{\mathbf{v}_0}{a} \sin \omega t \hat{\mathbf{a}} \]
The general pattern is:

\[ E = \frac{v_0}{2} \sin 2 \theta \left[ \left( \frac{\omega r}{\omega c} \right)^2 + \left( \frac{\omega r}{\omega c} \right)^4 - \left( \frac{\omega r}{\omega c} \right)^6 + \ldots \right] \]

The function in brackets is the Bessel function \( J_0(r) \).

Its plot looks like:

![Graph of Bessel function](image)
The Jumping ring

I = I_o \sin (\omega t - \theta(t))

Using Ampere's Law on the square loop of side L depicted above, we find:

\[ \delta B \cdot dl = M \oint_\gamma \mathbf{I} \cdot d\mathbf{l} = MNL I \]

The two horizontal contributions to the line integral cancel and the vertical contribution at \( r = \infty \) vanishes. This leaves

\[ B_z L + B_r \ell - B_r \ell + 0 = MN \ell I \]

\[ B_z = MN \frac{I_0}{\ell} \sin (\omega t - \theta) \]

\[ B_r \text{ undetermined} \]

\[ \mathbf{B} = MN \frac{I_0}{\ell} \sin (\omega t - \theta) \hat{z} + B_r \hat{r} \]
Using Gauss' Law on the Gaussian surface depicted above,

\[ \oint \mathbf{B} \cdot \mathbf{n} \, da = 0 \]

\[ B_r (2\pi r) \, dz + \pi r^2 (B_z (z+dz) - B_z (z)) = 0 \]

\[ B_r = \frac{-\pi r^2}{2\pi r^2} \frac{dB_z}{dz} \]

\[ B_r = -\frac{1}{2} \frac{dB_z}{dz} \quad \text{plug in } B_z \text{ from (a)} \]

\[ B_r = \frac{5}{2} \mu_0 I_0 \frac{d}{dz} \cos \omega t - \theta \]

\( B_r \) = radius of solenoid
\( r \) = radius of ring
Flux is contained in solenoid

\( \mathcal{E}_r = -\frac{d}{dt} \oint \mathbf{B} \cdot \mathbf{n} \, da \)

\[ = -\frac{d}{dt} (\pi a^2 B_z) \]

\[ = -\pi a^2 \frac{d}{dt} (\mu_0 I_0 \sin(\omega t - \theta)) \]

\( \mathcal{E}_r = -\pi a^2 \mu_0 I_0 \omega \cos(\omega t - \theta) \quad \text{clockwise, viewed from above} \)

\[ = \left( \pi a^2 \mu_0 I_0 \omega \cos(\omega t - \theta) \right) \quad \text{cw from above} \]
Kirchhoff's Law reduces to Ohm's law for this simple loop, and we have

\[ I = \frac{E}{R} \]

The Lorentz force on the ring (radius \( b \)) is

\[ \vec{F} = q \vec{v} \times \vec{B} \]

\[ = \frac{q \vec{v} \partial b \phi}{R} \]

\[ \int \vec{F} \cdot d\vec{l} = \frac{E R}{k} \text{ is counterclockwise} \]

or equivalently in the \( \hat{z} \) direction:

\[ d\vec{l} = b \, d\theta \hat{\phi} \]

\[ \int \vec{F} \cdot d\vec{l} = \frac{E b \phi}{R} \int_{0}^{\pi} d\theta \hat{\phi} \times (B_z \hat{r} + B_r \hat{\phi}) \]

Directions: From the figure,

\[ \hat{\phi} \times \hat{z} = \hat{r} \]

\[ \hat{\phi} \times \hat{r} = -\hat{\phi} \]

\[ \vec{F} = \frac{E b}{R} \oint_{\partial \Omega} \hat{\phi} \times (B_z \hat{r} + B_r \hat{\phi}) \]

\[ = \frac{2 \pi b}{R} \int_{0}^{\pi} \phi \, d\theta \]

\[ = \frac{2 \pi b}{R} \left[ \int_{0}^{\pi} \phi \, d\theta \right] (\pi a^2 \mu I_0 \cos(\alpha \theta)) (B_z \hat{r} - B_r \hat{\phi}) \]
\( P(\theta) = P(\theta) \) (cont.)

\[
\vec{F} = \frac{2\pi a^2 b (mnI_0)^2 \cos(wt-\theta)(\sin(wt-\theta) \hat{r} + \frac{b}{a} \cos^2(wt-\theta) \hat{\theta})}{R}
\]

\[
\vec{F} = \frac{2\pi a^2 b (mnI_0)^2 \cos(wt-\theta) \sin(wt-\theta) \hat{r} + \frac{b}{a} \cos^2(wt-\theta) \hat{\theta}}{R}
\]

The part of the force from the radial part of the field, \( B \cdot \vec{r} \), is the second term above.

\[
\vec{F} = \frac{2\pi a^2 b^2 \omega (mnI_0)^2 \cos^2(wt-\theta) \frac{d\theta}{\varphi}}{R}
\]

\( \hat{r} \) The average force on the ring in the \( \hat{r} \) direction is obtained by integrating the force from part (a) over a period:

\[
\langle F \rangle = \frac{1}{T} \int_0^T dt \vec{F}
\]

\[
= \frac{2\pi a^2 b^2 \omega (mnI_0)^2 \theta}{R} \int_0^{2\pi/\omega} \cos^2(wt-\theta) dt \hat{r}
\]

To do this integral, use the following variable substitution:

\[
\begin{align*}
\hat{r} &= \omega t - \theta \\
\hat{\theta} &= \omega t \\
u &= \omega t - \theta \\
u(0) &= -\theta \\
u(\pi/\omega) &= 2\pi - \theta \\
\end{align*}
\]

\[
\text{Integral} = \frac{1}{2\pi/\omega} \int_{\theta}^{2\pi - \theta} \cos^2 u \, du
\]
To do this last integral, use the following trick:

The integral of $\cos^2 u$ is the same as the integral of $\sin^2 u$ over a period. Hence,

$$\int_{\omega - \theta}^{\omega + \theta} \cos^2 u \, du = \frac{1}{2} \int_{\omega - \theta}^{\omega + \theta} \cos u \, du$$

$$= \frac{1}{2} \left[ \sin u \right]_{\omega - \theta}^{\omega + \theta}$$

$$= \frac{1}{2} \left( \sin \omega + \theta - \sin \omega - \theta \right)$$

$$= \frac{\sin \omega + \theta - \sin \omega - \theta}{2}$$

Hence the time-averaged force on the ring is

$$\langle \vec{F} \rangle = \frac{\pi a^2 b^2}{12} (\omega n \lambda)^2 \frac{d\theta}{d\omega}$$