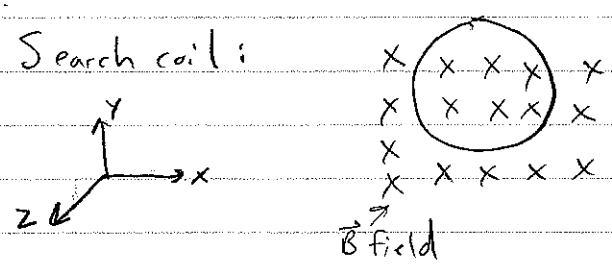
Ph262 PS 1 Solutions

YF E29.3 Search coils and credit cards

Search coil:



Area: A
Turns: N
Direction of area: $\hat{n}$ into page
$(\hat{n} = -\hat{z})$

Initial $\vec{B}$ field: $\vec{B}_i = B\hat{z}$ (out of page)
Final $\vec{B}$ field: $\vec{B}_f = 0$
Time for $\vec{B}$ change: Δt

(a) Faraday's Law:

$$\mathcal{E} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,da = -\frac{d\Phi_B}{dt}$$

Applied to this problem:

Initial flux $\Phi_B^i = \int \vec{B}\cdot\hat{n}\,da$

Total area is NA

$$= B\hat{z}\cdot(-\hat{z})(NA) = -BNA$$

Final flux $\Phi_B^f = 0$

Emf: $\mathcal{E} = -\frac{\Delta\Phi_B}{\Delta t} = \frac{BNA}{\Delta t}$

Emf direction: Clockwise (↻)
(By right hand rule around $\hat{n}$)

Lenz' Law check: Current creates $\vec{B}$ field to oppose change in external $\vec{B}$ field — OK ✓

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YF E29.3 (cont.)

Ohm's Law: $\mathcal{E} = IR$

Total charge that flows through coil in time Δt

$$dQ = I\,dt$$

$$\int_{Q=0}^{Q} dQ = \int_{t=0}^{\Delta t} I\,dt$$

$$Q = I\Delta t$$

Relating Q to emf $\mathcal{E}$:

$$\mathcal{E} = \frac{BNA}{\Delta t} = IR = \frac{QR}{\Delta t}$$

(a) $$\boxed{Q = \frac{BNA}{R}}$$

(b) A credit card reader <u>is</u> a search coil. However, instead of it moving, the card moves through it. It measures the $\vec{B}$ field, the magnitude of which, by part (a), is proportional to the charge on the card. The information is stored in the charge.

(c) The data is stored in the charge Q, so it is independent of the time of the swipe.

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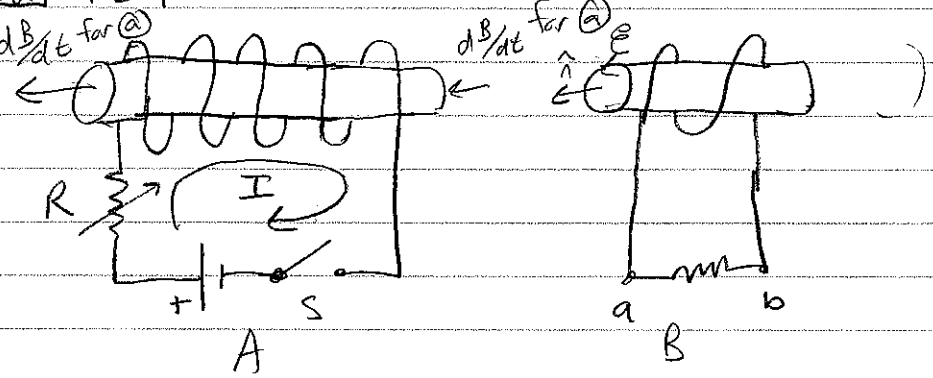
YF E29.17



ⓐ The switch S is opened after being closed.

Analysis:

The current I in the circuit initially flows as shown in the figure above. When S is opened, the current decreases to zero, so dI/dt goes counterclockwise, opposite to the original direction of I.

The helicity of the coils in the left circuit is such that a counterclockwise current in the circuit generates a $\vec{B}$ field in the solenoid pointing to the left. Hence, a changing current in this direction creates a changing $\vec{B}$ field in this direction. (See figure.)

The changing $\vec{B}$ field in the left circuit extends beyond that circuit into the region of the right circuit. This creates a changing flux through the coils of the right circuit, such that the flux going to the left through this area is increasing. ($A\,d\vec{B}/dt \cdot \hat{n} > 0$ in the figure.)

Lenz' Law: The changing flux in the right circuit creates an emf $\mathcal{E}$ around $\hat{n}$ (by the right hand rule, counterclockwise when viewed from the left) that opposes the time change of this flux!

$$\mathcal{E} = -\frac{d\Phi}{dt} = -A\frac{dB}{dt} \quad \text{counterclockwise}$$
$$= A\frac{dB}{dt} \quad \text{clockwise.}$$

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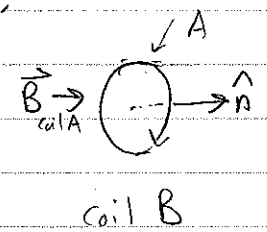
YF E29.17 (a) (cont.)

By Ohm's Law, the emf $\mathcal{E}$ creates a current I in the same direction by $\mathcal{E} = IR$. Given the helicity of the coils in the right circuit, this current circulates counterclockwise in the right circuit as viewed from above.

Therefore, $\boxed{\text{the induced current flows from } a \text{ to } b.}$

(b) Coil B is brought closer to coil A with the switch closed.

By the helicity of the winding in coil A, the $\vec{B}$ field from coil A points to the right. As coil B is brought closer to coil A, it experiences and increasing $\vec{B}$ field to the right, and correspondingly an increasing magnetic flux to the right.



$$A \frac{d\vec{B}}{dt} \cdot \hat{n} > 0$$

Lenz' Law → By Faraday's Law, an emf $\mathcal{E}$ around $\hat{n}$ (by the right hand rule, clockwise when viewed from the left) is created that opposes the time change of this flux.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -A\frac{dB}{dt} \quad \text{clockwise}$$

$$= A\frac{dB}{dt} \quad \text{counter clockwise}$$

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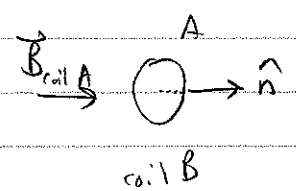
YF E29.17 (b) (cont.)

By Ohm's Law, the emf creates a current I in the same direction by $\mathcal{E} = IR$. Given the helicity of coil B, this current circulates <u>clockwise</u> in circuit B as viewed from above.

Therefore, $\boxed{\text{the induced current flows from } \underline{b \text{ to } a}.}$

(c) The resistance R in circuit A is decreased while the switch remains closed.

As R is decreased, more current flows through circuit A by Ohm's Law, $\mathcal{E} = IR$. This increased current increases the magnetic field in coil B, which points to the <u>right</u>. This increases the magnetic flux through coil B to the <u>right</u>.



$$A \frac{d\vec{B}}{dt} \cdot \hat{n} > 0$$

This is the same situation as in part (b), so the same analysis applies, and therefore

$$\boxed{\text{the induced current flows from } \underline{b \text{ to } a}}$$

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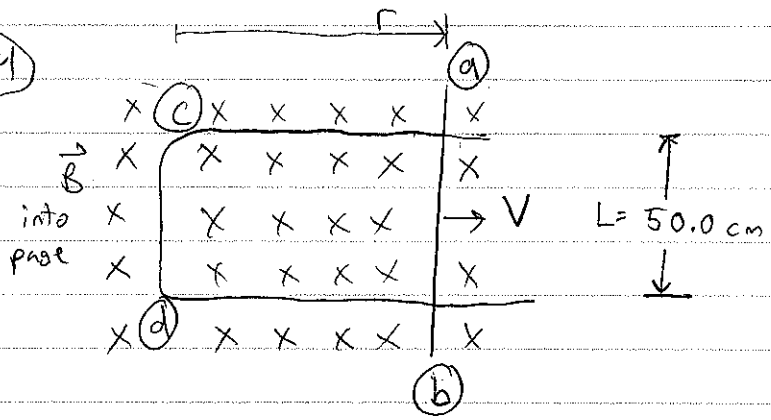
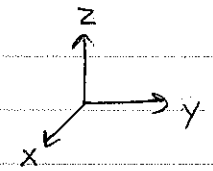
YF E29.24





coordinate system

$\vec{B} = 0.800\,T$ into page
$= 0.800\,T\,(-\hat{x})$

$\vec{v} = 7.50\,m/s\ \hat{y}$

(a) Find the emf in the rod.

Method 1: "motional emf" (I dislike this method.)

$$|\mathcal{E}| = |vBL| = (7.50\,m/s)(0.800\,T)\left(\frac{50.00\,cm}{}\cdot\frac{m}{100\,cm}\right)$$

$$\boxed{|\mathcal{E}| = 3.00\,V}$$

(b) Notice that Method 1 doesn't help you figure out which <u>direction</u> the emf is in. That's a big part of why this method is lousy. A better way to do part (a) that will also answer the direction question here, is the following

Method 2:

Faraday's Law: $\mathcal{E} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,da$

$\hat{n}$: into page $(-\hat{x})$

$\vec{B}\cdot\hat{n} = B(-\hat{x})\cdot(-\hat{x}) = B$

$\Phi = \int \vec{B}\cdot\hat{n}\,da = BLr$

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YF E29.24 (b) (cont.)

$$\mathcal{E} = \frac{-d\Phi}{dt} = -BL\frac{dr}{dt} = -BLV$$

Direction: right hand rule around $\hat{n}$ is clockwise

$$\Rightarrow \mathcal{E} = -BLV \text{ clockwise}$$

$$= BLV \text{ counterclockwise}$$

If the wire is ohmic, $\mathcal{E} = IR$, so

$$\boxed{I = \frac{BLV}{R} \quad \underline{\text{counterclockwise}}}$$

(only the direction was asked for, but Faraday's law gives so much more!)

(c) If $R = 1.50\,\Omega$ (assumed constant) find $\vec{F}$ needed to keep rod moving at $\vec{v}$

The Lorentz force on the rod is

$$\vec{F} = I\vec{L} \times \vec{B}, \quad \text{where } \vec{L} = L\hat{z}, \text{ in the direction of the current}$$

$$\vec{F} = \underbrace{\left(\frac{BLV}{R}\right)}_{\text{from part (b)}} L\hat{z} \times B(-\hat{x})$$

$$= \frac{B^2L^2V}{R}\hat{z} \times (-\hat{x}) \qquad \hat{z} \times \hat{x} = \hat{y}$$

$$\boxed{\vec{F} = \frac{B^2L^2V}{R}\hat{y}}$$

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YF E29.24 c (cont.)

Plugging in numbers,

$$\vec{F} = \frac{(0.800\,T)^2\left(\frac{50.00\,cm}{1}\cdot\frac{m}{100\,cm}\right)^2(7.5\,m/s)}{1.50\,\Omega}\,\hat{y}$$

$$\boxed{\vec{F} = 0.800\,N\,\hat{y}}$$

d) Rate of mechanical work:

from part c

$$P_{mech} = \frac{dW}{dt} = \vec{F}\cdot\vec{v} = \left(\frac{B^2L^2v}{R}\hat{y}\right)\cdot(v\hat{y})$$

$$\boxed{P_{mech} = \frac{B^2L^2v^2}{R}}$$

Plugging in numbers:

$$P_{mech} = (0.800\,N)(7.50\,m/s)$$

$$\boxed{P_{mech} = 6.00\,W}$$

Rate of ohmic heating

from part b

$$P_{ohmic} = I^2R = \left(\frac{BLv}{R}\right)^2R$$

$$\boxed{P_{ohmic} = \frac{B^2L^2v^2}{R}}$$

Algebraically, $\boxed{P_{mech} = P_{ohmic}}$

Hence, $\boxed{P_{ohmic} = 6.00\,W}$

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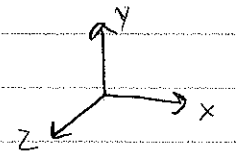
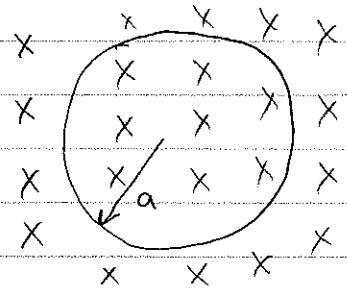
YF P29.47 Self-emf





Wire Loop: radius a, resistance R

$$\vec{B}^{ext}_{init} = -B^{ext}_{init}\,\hat{z}$$

$$\vec{B}^{ext}_{final} = 0$$

$$\hat{n} \text{ (direction of loop area)} = -\hat{z}$$

(a) When the external $\vec{B}$ drops to zero, it creates an emf in the loop by Faraday's Law:

$$\mathcal{E}_{loop} = -\frac{d}{dt}\int \vec{B}^{ext}\cdot\hat{n}\,da$$

$$= \frac{-(\vec{B}^{ext}_{fin}\cdot\hat{n} - \vec{B}^{ext}_{init}\cdot\hat{n})\,\pi a^2}{\Delta t}$$

area of loop

$$= -\left(0 - (-B^{ext}_{init}\,\hat{z}\cdot(-\hat{z}))\right)\frac{\pi a^2}{\Delta t}$$

$$= \left(B^{ext}_{init}\,\pi a^2/\Delta t\right) \text{ clockwise}$$ (right hand rule around $\hat{n}$)

For an ohmic wire, the current goes in the same direction as $\mathcal{E}$.

Approximating the field from this wire to be uniform in the loop, we find

$$\vec{B}_{loop} = \frac{-\mu_0 i}{2a}\,\hat{z}$$

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YF P29.47 (a) (cont.)

The flux of this field through the loop is

$$\Phi_{loop} = \int \vec{B}_{loop} \cdot \hat{n}\, da$$

$$= -\frac{\mu_0 i}{2a} \hat{z} \cdot (-\hat{z}) \pi a^2$$

$$\boxed{\Phi_{loop} = \frac{\pi a \mu_0 i}{2} \text{ into the page}}$$ (in $-\hat{z}$ direction)

(b) By Faraday's Law,

$$\mathcal{E}_{loop} = -\frac{d\Phi_{loop}}{dt}$$

and by Ohm's Law, $\mathcal{E}_{loop} = iR$.

Putting these together,

$$iR = -\frac{d}{dt}\left(\frac{\pi a \mu_0 i}{2}\right)$$

$$= -\frac{\pi a \mu_0}{2} \frac{di}{dt}$$

$$\Rightarrow \boxed{\frac{di}{dt} = -\left(\frac{2R}{\pi \mu_0 a}\right) i}$$

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YF P29.47 (c) If $i(t=0) = i_0$, then we can solve this differential equation:

$$\frac{di}{dt} = -\left(\frac{2R}{\pi \mu_0 a}\right) i$$

$$\frac{di}{i} = -\left(\frac{2R}{\pi \mu_0 a}\right) dt$$

$$\int_{i=i_0}^{i} \frac{di}{i} = -\left(\frac{2R}{\pi \mu_0 a}\right) \int_{t=0}^{t} dt$$

$$\ln\left(\frac{i}{i_0}\right) = -\left(\frac{2R}{\pi \mu_0 a}\right) t$$

Exponentiate both sides, use $e^{\ln x} = x$:

$$\frac{i}{i_0} = e^{-(2R/\pi \mu_0 a)t}$$

$$\boxed{i = i_0 e^{-(2R/\pi \mu_0 a)t}}$$

(d) For $a = 50$ cm, $R = 0.10\,\Omega$, when is $i = 0.010\, i_0$?

To solve this, we take the natural log of the solution in part (c), or more simply just use this formula

$$t = -\left(\frac{\pi \mu_0 a}{2R}\right) \ln\left(\frac{0.010\, i_0}{i_0}\right)$$

$$= -\left(\frac{\pi (4\pi \times 10^{-7}\ \mathrm{Wb/A\cdot m})\left(\frac{50\ \mathrm{cm}}{1}\,\frac{\mathrm{m}}{100\ \mathrm{cm}}\right)}{0.10\,\Omega}\right) \ln(0.010)$$

$$\boxed{t = 4.55 \times 10^{-5}\ \mathrm{s}}$$ fast!

(e) We ignore this effect because we are interested in physics on a much slower time scale than 50 μs.

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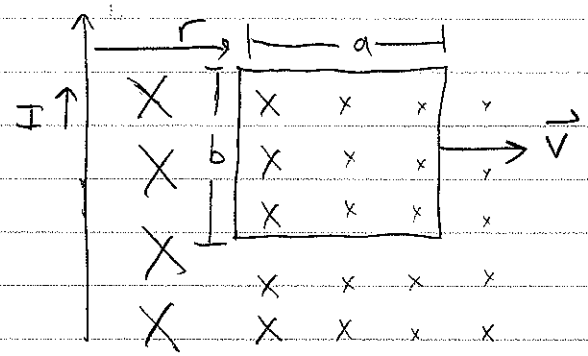
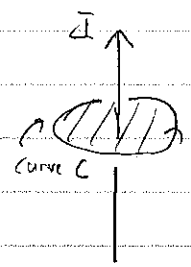
YF 29.49



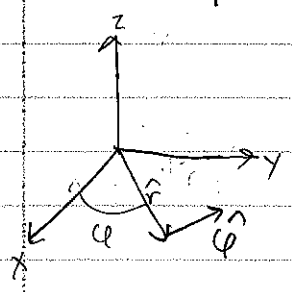
(a) Find $\mathcal{E}_{loop}$ using (i) Faraday's Law, (ii) motional emf

To begin, the current in the wire creates a $\vec{B}$ field circulating around it whose magnitude drops off like $1/r$. In the plane of the loop, I have indicated this by using decreasingly large x's. To find this field, we use Ampere's Law:



$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{in}$$

$$\int_{\varphi=0}^{2\pi} B_\varphi(r)\hat{\varphi}\cdot r\,d\varphi\,\hat{\varphi} = \mu_0 I$$



$$r B_\varphi(r) \int_{\varphi=0}^{2\pi} \cancel{\hat{\varphi}\cdot\hat{\varphi}}^{\,1}\, d\varphi = \mu_0 I$$

$$2\pi r B_\varphi(r) = \mu_0 I$$

$$B_\varphi(r) = \frac{\mu_0 I}{2\pi r}$$

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\varphi}$$ (circulates ccw, viewed from above)

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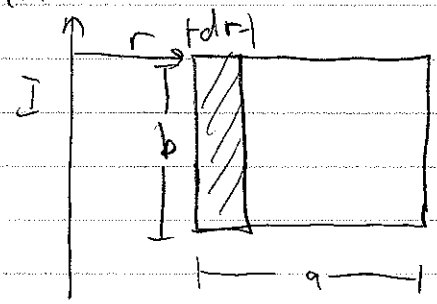
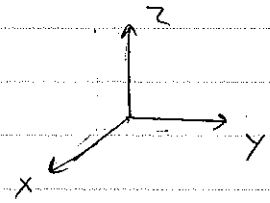
YF 29.49 (a) (cont.)

To use Faraday's law, first we calculate the flux through the loop, then we calculate how it changes when it is moving with velocity v.

Because $\vec{B}$ is not constant through the loop, we divide the area into infinitesimally thin strips of width dr — so thin in fact that $\vec{B}$ is constant in each strip. We then add the flux through each strip to find the total flux:



$\hat{n}$: choose to be into page $(-\hat{x})$



$$d\Phi = \vec{B}_{wire} \cdot \hat{n}\, da$$

$$= \left(\frac{\mu_0 I}{2\pi r}(-\hat{x})\right) \cdot (-\hat{x})\,(b\,dr)$$

$$= \frac{\mu_0 I}{2\pi r}\, b\,dr$$

$$\Phi = \int_{r=r}^{r+a} \left(\frac{\mu_0 I b}{2\pi r}\right) dr$$

$$\Phi = \left(\frac{\mu_0 I b}{2\pi}\right) \ln\left(\frac{r+a}{r}\right) \text{ into page}$$

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YF 29.49 (a) (still cont.)

The emf in the loop is then calculated by Faraday's Law, with a judicious application of the chain rule for taking derivatives:

$$\mathcal{E} = \frac{-d\Phi}{dt} = -\frac{d\Phi}{dr}\frac{dr}{dt}$$

$$= -\left(\frac{\mu_0 I b}{2\pi}\right)\frac{d}{dr}\left(\ln(r+a) - \ln r\right)\cancel{\frac{dr}{dt}}\ v$$

$$= -\left(\frac{\mu_0 I b v}{2\pi}\right)\left(\frac{1}{r+a} - \frac{1}{r}\right)$$

$$= \left(\frac{\mu_0 I b v}{2\pi}\right)\left(\frac{1}{r} - \frac{1}{r+a}\right)$$

$$= \left(\frac{\mu_0 I b v}{2\pi}\right)\left(\frac{\cancel{r}+a-\cancel{r}}{r(r+a)}\right)$$

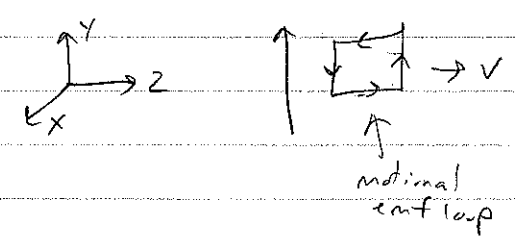
(a) (i) $$\boxed{\mathcal{E} = \frac{\mu_0 I b v a}{2\pi r(r+a)} \quad \text{clockwise}}$$

direction is by right hand rule around $\hat{n}$

(only magnitude requested, but we can already tell the direction)

(ii) Using the motional emf argument, (I dislike this method)

$$\mathcal{E} = \oint(\vec{v}\times\vec{B})\cdot d\vec{l}$$



$$\vec{v}\times\vec{B} = v\hat{z}\times B(r)(-\hat{x})$$
$$= vB(r)\,\hat{z}\times(-\hat{x})$$
$$= vB(r)\,\hat{y}$$

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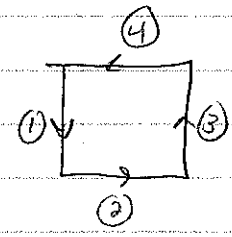
YF 29.49 (a) (ii) Doing the line integral in four pieces:



$$\oint (\vec{v}\times\vec{B})\cdot d\vec{\ell} = vB(r)\,\hat{y}\cdot b(-\hat{y}) \quad (1)$$

$$+ \int_{r=r}^{r+a} vB(r)\,dr\,\hat{y}\cdot\hat{z} \quad (2)$$

$$+ vB(r+a)\,\hat{y}\cdot b(\hat{y}) \quad (3)$$

$$+ \int_{r=r+a}^{r} vB(r)\,dr\,\hat{y}\cdot(-\hat{z}) \quad (4)$$

Terms (2) and (4) cancel, leaving

$$\mathcal{E} = vb\left(B(r+a) - B(r)\right)$$

$$= \frac{\mu_0 v b I}{2\pi}\left(\frac{1}{r+a} - \frac{1}{r}\right)$$

$$= \frac{\mu_0 v b I}{2\pi}\left(\frac{r-(r+a)}{r(r+a)}\right)$$

$$\mathcal{E} = -\frac{\mu_0 I b v a}{2\pi r(r+a)}$$ counterclockwise (direction of line integral I chose)

$$\boxed{\mathcal{E} = \frac{\mu_0 I b v a}{2\pi r(r+a)} \quad \text{clockwise}}$$ (Same as (a) (i))

(b) (i) We already found the emf direction, and ohmic materials have the current moving in the same direction.
The intuition of "Lenz' Law" agrees with this:
a clockwise current will create a $\vec{B}$ field into the page, compensating for the loss of $\vec{B}$ field through the loop that occurs as the loop moves away from the wire.

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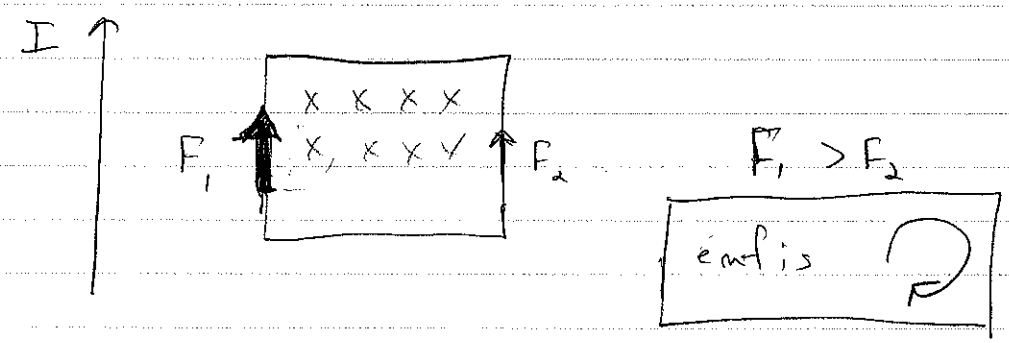
Y F 29.49 (b) (ii) Using the Lorentz force formula,

$$\vec{F} = I\vec{L} \times \vec{B},$$

The only force along the direction of the wire occurs in the two vertical parts of the loop (of length b). Because the $\vec{B}$ field is larger on the part closer to the wire, the net force is given by the force here, which generates a clockwise emf. See the figure below.



(c) Sanity checks:

i) If $v=0$, the emf is

$$\boxed{\mathcal{E} = \lim_{v\to 0} \frac{\mu_0 I b v a}{2\pi r(r+a)} = 0} \quad \checkmark$$

No motion, no emf

ii) If $a \to 0$, the emf is

$$\boxed{\mathcal{E} = \lim_{a\to 0} \frac{\mu_0 I b v a}{2\pi r(r+a)} = 0} \quad \checkmark$$

No area, no flux

iii) If $r \to \infty$, the emf is

$$\boxed{\mathcal{E} = \lim_{r\to\infty} \frac{\mu_0 I b v a}{2\pi r(r+a)} = 0} \quad \checkmark$$

No $\vec{B}$, no flux

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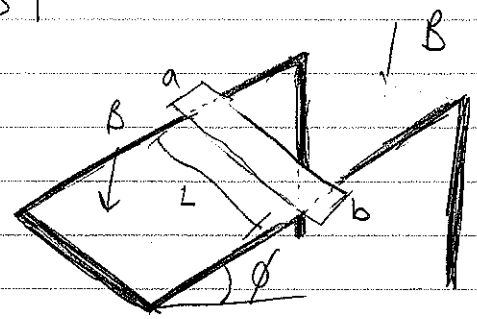
YF 29.76

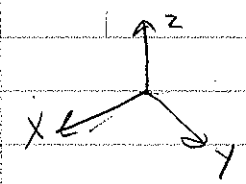


Bar: length L
mass M
resistance R

Frictionless inclined metal rails of negligible resistance.

Uniform $\vec{B}$ downwards

Coordinate system:



(a) Bar slides down: Flux downwards through closed circuit decreases, so by Lenz' law current will be generated to make a $\vec{B}$ field pointing down, so current flows clockwise around the loop as viewed from above.

$$\boxed{\text{Hence, the current goes from a to b.}}$$

(b) To find the terminal velocity, we use Newton's 2nd Law on the bar

$$\sum_i \vec{F}_i^{bar} = M_{bar}\vec{a}_{bar}$$

$$\vec{F}_{Normal} + \vec{F}_g + \vec{F}_{Lorentz} = M_{bar}\vec{a}_{bar} = 0 \quad \text{at terminal velocity}$$

$$\begin{aligned}\vec{F}_{Lorentz} &= I\vec{\ell} \times \vec{B} \\ &= I L \hat{y} \times B(-\hat{z}) \\ &= -ILB\hat{x}\end{aligned}$$

Ohm's Law: $I = \mathcal{E}/R$ in bar, so

$$\vec{F}_{Lorentz} = -\frac{\mathcal{E}LB}{R}\hat{x}$$

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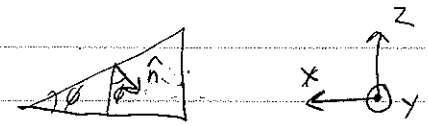
(b) (cont.)

YF 29.76 We use Faraday's Law to calculate the emf

$$\mathcal{E} = -\frac{d}{dt}\int \vec{B}\cdot\hat{n}\,da$$



Must choose a direction for $\hat{n}$ $\perp$ to area.

Choose $\hat{n} = -\cos\phi\,\hat{z} - \sin\phi\,\hat{x}$

$$\frac{da}{dt} = \frac{L\,dr}{dt} = Lv$$

$$\vec{B}\cdot\hat{n} = -B\,\hat{z}\cdot(-\cos\phi\,\hat{z} - \sin\phi\,\hat{x})$$

$$= B\cos\phi$$

$$\mathcal{E} = \int \vec{B}\cdot\hat{n}\,\frac{da}{dt}$$

from right hand rule around $\hat{n}$.

$\mathcal{E} = BLv\cos\phi$ clockwise, viewed from above (agrees w/ Lenz' Law in (a))

The Lorentz force is therefore

$$\vec{F}_{Lorentz} = -\frac{B^2L^2v\cos\phi}{R}\,\hat{x}$$

The gravitational force is $\vec{F}_g = -mg\hat{z}$

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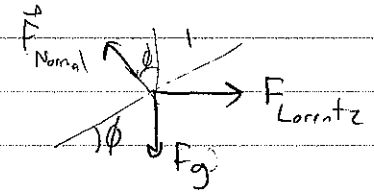
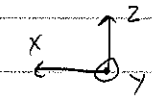
YF 29.76 (b) (still cont.)

Drawing a free-body diagram:





Coord system

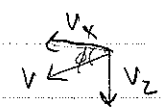
$$\sum F_z = 0 \quad : \quad F_N \cos\phi - mg = 0$$

$$F_N = \frac{mg}{\cos\phi}$$

$$\sum F_x = 0 \quad : \quad F_N \sin\phi - \frac{B^2 L^2 v}{R}\cos\phi = 0$$

$$\left(\frac{B^2L^2}{R}\cos\phi\right) v = \left(\frac{mg}{\cos\phi}\right)\sin\phi$$

$$\boxed{v = \frac{mgR}{B^2L^2}\,\frac{\tan\phi}{\cos\phi}}$$



As a vector, $\boxed{\vec{v} = v\cos\phi\,\hat{x} - v\sin\phi\,\hat{z}}$

(c) The induced current at terminal velocity can be found using the emf formula in (b) and Ohm's Law, $\mathcal{E} = IR$:

$$\mathcal{E} = BLv\cos\phi \quad \text{clockwise from above}$$

$$I = \frac{\mathcal{E}}{R} = \frac{BLv}{R}\cos\phi = \left(\frac{BL\cancel{\cos\phi}}{\cancel{R}}\right)\left(\frac{mg\cancel{R}}{B^2L^2}\right)\frac{\tan\phi}{\cancel{\cos\phi}}$$

$$\boxed{I = \frac{mg}{BL}\tan\phi \quad \text{clockwise as viewed from above}}$$

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YF 29.76

(d) Electrical energy is converted into thermal energy by Ohmic I^2R heating. At terminal velocity this rate is

$$P = I^2 R$$

$$\boxed{P = \frac{m^2 g^2 R}{B^2 L^2} \tan^2\phi}$$

(e) The rate of work being done on the bar by gravity is

$$\frac{dW_g}{dt} = \vec{F}_g \cdot \vec{v}$$

$$= (-mg\hat{z}) \cdot (v\cos\phi\,\hat{x} - v\sin\phi\,\hat{z})$$

$$= mgv\sin\phi$$

$$= (mg\sin\phi)\left(\frac{mgR}{B^2L^2}\right)\frac{\tan\phi}{\cos\phi}$$

$$\boxed{\frac{dW_g}{dt} = \frac{m^2 g^2 R}{B^2 L^2} \tan^2\phi}$$

The same as d. Gravity is causing the I^2R heating, not the Lorentz force!

(This makes sense because we know that magnetic fields can do no work.)

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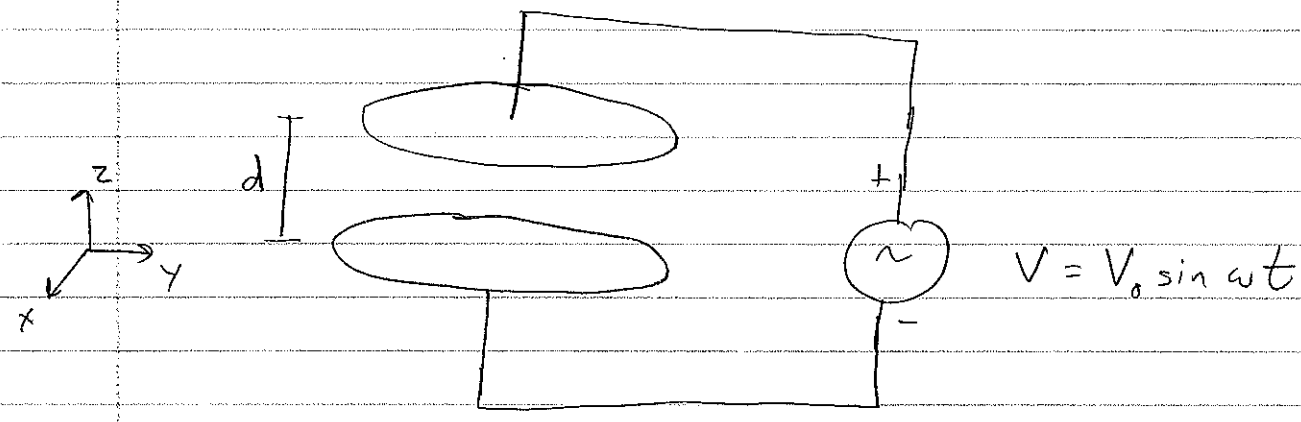
I High frequency capacitors



a To first approximation, no $\vec{B}$ fields exist in the capacitor, and Faraday's Law says:

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot \hat{n} \, da = 0$$

This means that the electric field is <u>conservative</u> to first approximation, so that the electric field is expressible as (minus) the gradient of the potential:

$$\vec{E}_1 = -\nabla V$$

$$= -\frac{V_{top} - V_{bottom}}{\Delta z}$$

$$\boxed{\vec{E}_1 = \frac{-V_0}{d} \sin \omega t \, \hat{z}}$$

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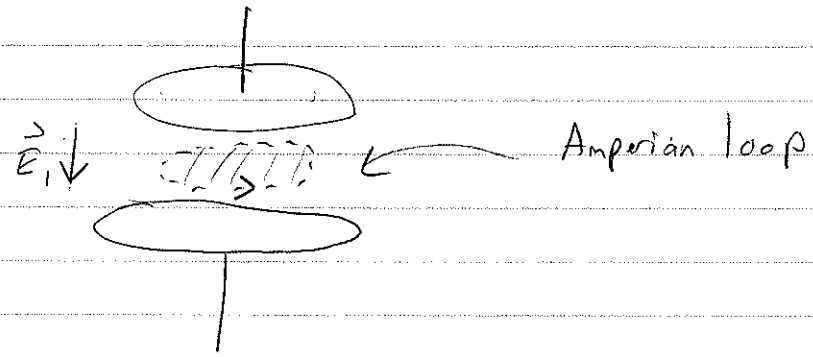
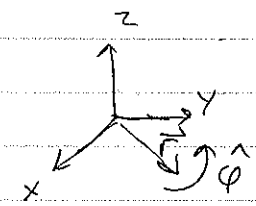
I (b)





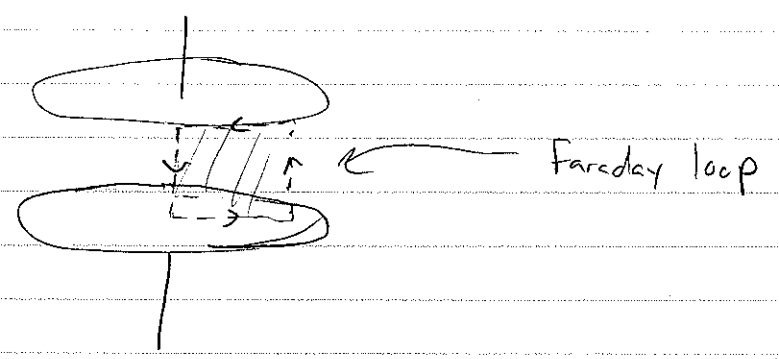
Applying the Ampere-Maxwell law to a loop inside the capacitor and coaxial with it,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \varepsilon_0 \frac{d}{dt}\int \vec{E}_1\cdot\hat{n}\, da$$

$$2\cancel{\pi} r B_1 = \mu_0 \varepsilon_0 \cancel{\pi} r^{\cancel{2}} \frac{V_0}{d}\frac{d}{dt}(\sin\omega t) \qquad \text{use } \mu_0\varepsilon_0 = \frac{1}{c^2}$$

$$\boxed{\vec{B}_1 = -\frac{\omega r}{2c^2}\left(\frac{V_0}{d}\right)\cos\omega t\ \hat{\varphi}}$$ (clockwise, viewed from above)

(c)



Applying Faraday's law to the loop shown, we can find $\vec{E}_2$:

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}_1\cdot\hat{n}\, da$$

Using $\vec{E} = \vec{E}_1 + \vec{E}_2$, we see the line integral of $\vec{E}_1$ vanishes from part (a). There is no correction at the center, so $E_2 = 0$ at $r=0$. Moreover, the top and bottom parts of this line integral cancel by symmetry.

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(I) (c) (cont.) Faraday's Law then gives:

$$\oint \vec{E}\cdot d\vec{l} = E_z\,\hat{z}\cdot(\hat{z}d)$$

$$= E_z d$$

$$= -\frac{d}{dt}\int \vec{B}_1\cdot\hat{n}\,da$$

$$\int \vec{B}_1\cdot\hat{n}\,da = \int\left(-\frac{\omega r}{2c^2}\right)\left(\frac{V_0}{d}\right)\cos\omega t\;\cancel{\hat{\varphi}\cdot\hat{\varphi}}^{\,1}\,d\,dr$$

$$= -\left(\frac{\omega}{2c^2}\right)V_0\cos\omega t\int r\,dr$$

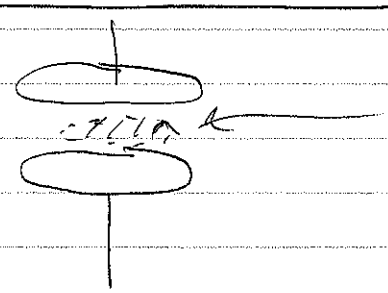
$$= -\left(\frac{\omega r^2}{4c^2}\right)V_0\cos\omega t$$

$$E_z d = \frac{-d}{dt}\int\vec{B}\cdot\hat{n}\,da$$

$$\vec{E}_z = -\left(\frac{\omega r^2}{4c^2}\right)\left(\frac{V_0}{d}\right)(-\omega\sin\omega t)\,\hat{z}$$

$$\boxed{\vec{E}_z = \left(\frac{\omega^2 r^2}{4c^2}\right)\left(\frac{V_0}{d}\right)\sin\omega t\,\hat{z}}$$

(d)



Applying the Ampere-Maxwell law to the same loop as in part (b),

$$\oint\vec{B}_2\cdot d\vec{l} = \frac{1}{c^2}\frac{d}{dt}\int\vec{E}_2\cdot\hat{n}\,da$$

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I (d) (cont.)

$$\cancel{2\pi} r B_2 = \frac{1}{c^2}\left(\frac{\omega^2}{4c^2}\right)\left(\frac{V_0}{d}\right) \int \cancel{r^2 (2\pi r\, dr)} \frac{\partial}{\partial t} \sin\omega t \quad \to \frac{1}{4} r^4$$

$$B_2 = \frac{\omega^2 r^3}{16 c^4}\left(\frac{V_0}{d}\right) \omega \cos\omega t$$

$$\boxed{\vec{B}_2 = \frac{\omega^3 r^3}{16 c^4}\left(\frac{V_0}{d}\right) \cos\omega t\, \hat{\varphi}}$$ (clockwise, viewed from above)

(e) By Faraday's law on the loop in (c):

$$\oint \vec{E}_3 \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B}_2 \cdot \hat{n}\, da$$

$$-E_3 d = \frac{\omega^3}{16 c^4} \frac{V_0}{d} \left[\int r^3 \cdot d\, dr\right] -\frac{d}{dt} \cos\omega t$$

$$E_3 = \frac{-\omega^3}{16 c^4} \frac{V_0}{d} \frac{1}{4} r^4\, \omega \sin\omega t$$

$$\boxed{\vec{E}_3 = \frac{-\omega^4 r^4}{64 c^4} \frac{V_0}{d} \sin\omega t\, \hat{z}}$$

(f) Writing out the series for $\vec{E}$ we have

$$\vec{E} = -\frac{V_0}{d} \sin\omega t\, \hat{z} \Big(\underbrace{1}_{E_1} \underbrace{- \frac{1}{2^2}\left(\frac{\omega r}{c}\right)^2}_{E_2} + \underbrace{\frac{1}{2^2 \cdot 4^2}\left(\frac{\omega r}{c}\right)^4}_{E_3} + \dots \Big)$$

If we worked out the next term, we would find

$$E_4 = \frac{1}{2^2 \cdot 4^2 \cdot 6^2}\left(\frac{\omega r}{c}\right)^6 \frac{V_0}{d} \sin\omega t\, \hat{z}.$$

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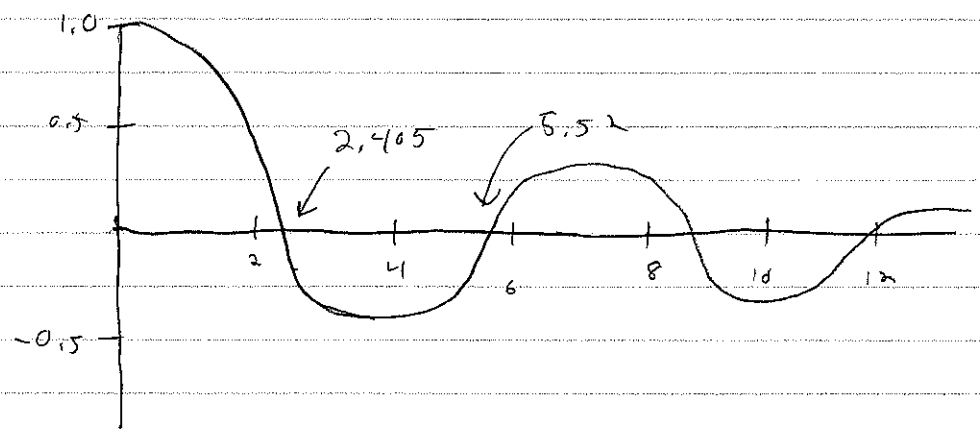
1 (f) (cont'd)

The general pattern is

$$\boxed{\vec{E} = -\frac{V_0}{d}\sin\omega t\,\hat{z}\left[1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega r}{2c}\right)^6 + \cdots\right]}$$

(g) The function in brackets is the Bessel function $\boxed{J_0(r)}$.
Its plot looks like



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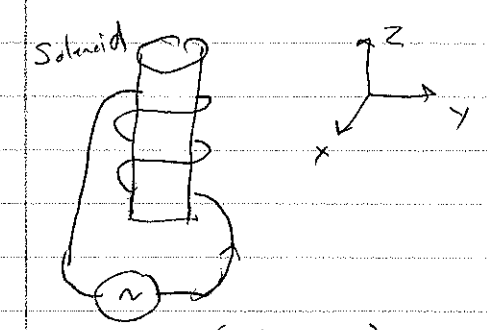
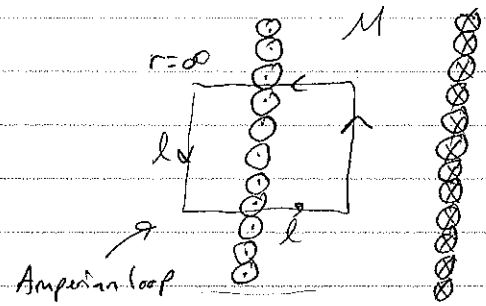
(II) The Jumping ring



$I = I_0 \sin(\omega t - \theta(z))$

n turns/length



Solenoid cross section

(a) Using Ampere's Law on the square loop of side l depicted above, we find

$$\oint \vec{B}\cdot dl = \mu I_{in} = \mu n l I$$

The two horizontal contributions to the line integral cancel and the vertical contribution at $r=\infty$ vanishes. This leaves

$$B_z l + \cancel{B_r l} - \cancel{B_r l} + 0 = \mu n l I$$

$$B_z = \mu n I_0 \sin(\omega t - \theta)$$

B_r undetermined

$$\boxed{\vec{B} = \mu n I_0 \sin(\omega t - \theta)\,\hat{z} + B_r \hat{r}}$$

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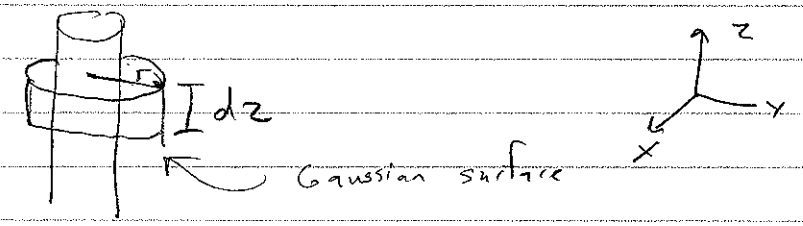
II (b)



Using Gauss' Law on the Gaussian surface depicted above,

$$\int \vec{B} \cdot \hat{n}\, da = 0$$

$$B_r (2\pi r) dz + \pi r^2 \left(B_z (z+dz) - B_z (z) \right) = 0$$

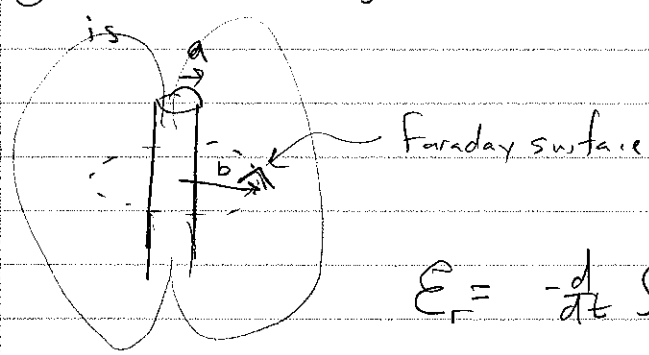
$$B_r = \frac{-\pi r^{\cancel{2}}}{2\cancel{\pi} r} \frac{dB_z}{dz}$$

$$\boxed{B_r = -\frac{r}{2} \frac{dB_z}{dz}}$$

plug in B_z from (a)

$$\boxed{B_r = \frac{r}{2} \mu n I_0 \frac{d\theta}{dz} \cos(\omega t - \theta)}$$

(c) The emf generated in an external coaxial ring



a = radius of solenoid
b = radius of ring
flux is contained in solenoid.

$$\mathcal{E}_r = -\frac{d}{dt} \int \vec{B} \cdot \hat{n}\, da$$

$$= -\frac{d}{dt} \left(\pi a^2 B_z \right)$$

$$= -\pi a^2 \frac{d}{dt} \left(\mu n I_0 \sin(\omega t - \theta) \right)$$

$$\boxed{\mathcal{E}_r = -\pi a^2 \mu n I_0 \omega \cos(\omega t - \theta)}$$ clockwise, viewed from above

$$= \left(\pi a^2 \mu n I_0 \omega \cos(\omega t - \theta) \right.$$ ccw from above $\left. \right)$

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7 (d) Kirchoff's Law reduces to Ohm's law for this simple loop, and we have

$$\boxed{I_r = \mathcal{E}_r / R}$$

(e) The Lorentz force on the ring (radius b) is

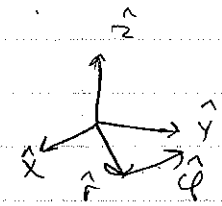
$$\vec{F} = q\vec{v} \times \vec{B}$$

$$= \oint I_r d\vec{\ell} \times \vec{B}$$

$I_r d\vec{\ell} = \frac{\mathcal{E}_r}{R} d\vec{\ell}$ is counterclockwise,

or equivalently in the $\hat{\varphi}$ direction:

$d\vec{\ell} = b\, d\varphi\, \hat{\varphi}$



$$\oint I_r d\vec{\ell} \times \vec{B} = \frac{\mathcal{E}_r b}{R} \oint d\varphi\, \hat{\varphi} \times (B_z \hat{z} + B_r \hat{r})$$

Directions: From the figure, $\hat{\varphi} \times \hat{z} = \hat{r}$

$\hat{\varphi} \times \hat{r} = -\hat{z}$

$$\vec{F} = \frac{\mathcal{E}_r b}{R} \underbrace{\oint d\varphi}_{2\pi} B_z \hat{r} - B_r \hat{z}$$

$$= \left(\frac{2\pi b}{R}\right) (\pi a^2 \mu n I_0 \omega) \cos(\omega t - \theta) \left(B_z \hat{r} - B_r \hat{z}\right)$$

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II

(e) (cont.)

$$\vec{F} = \frac{2\pi^2 a^2 b}{R}(\mu n I_0)^2 \omega \cos(\omega t-\theta)\left(\sin(\omega t-\theta)\,\hat{r} + \frac{b}{2}\frac{d\theta}{dz}\cos(\omega t-\theta)\,\hat{z}\right)$$

$$\boxed{\vec{F} = \frac{2\pi^2 a^2 b}{R}(\mu n I_0)^2 \omega \cos(\omega t-\theta)\left[\sin(\omega t-\theta)\,\hat{r} + \frac{b}{2}\frac{d\theta}{dz}\cos(\omega t-\theta)\,\hat{z}\right]}$$

The part of the force from the radial part of the field, $B_r\hat{r}$ is the second term above)

$$\boxed{\vec{F} = \frac{\pi^2 a^2 b^2 \omega}{R}(\mu n I_0)^2 \cos^2(\omega t-\theta)\frac{d\theta}{dz}\,\hat{z}}$$

(f) The average force on the ring in the $\hat{z}$ direction is obtained by integrating the force from part (e) over a period:

$$T = \frac{2\pi}{\omega}$$

$$\langle \vec{F} \rangle = \int_{t=0}^{T} dt\, \vec{F}$$

$$= \frac{\pi^2 a^2 b^2 \omega}{R}(\mu_0 n I_0)^2 \theta' \int_{t=0}^{2\pi/\omega} \cos^2(\omega t-\theta)\,dt\,\hat{z}$$

To do this integral, use the following variable substitution

$u = \omega t - \theta \qquad u(t=0) = -\theta$

$du = \omega\, dt \qquad u(t=2\pi/\omega) = 2\pi - \theta$

$$\text{integral} = \frac{1}{\omega}\int_{u=-\theta}^{2\pi-\theta} \cos^2 u\, du$$

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II (f) (cont.)

To do this last integral, use the following trick: The integral of $\cos^2 u$ is the same as the integral of $\sin^2 u$ over a period. Hence,

$$\text{integral} = \frac{1}{\omega}\int_{u=-\theta}^{2\pi-\theta} \cos^2 u\, du = \frac{1}{\omega}\int_{u=0}^{2\pi} \cos^2 u\, du$$

$$= \frac{1}{\omega}\int_{u=0}^{2\pi} \frac{1}{2}\left(\sin^2 u + \cos^2 u\right) du$$

$$= \frac{1}{\omega}\frac{1}{2}\int_{u=0}^{2\pi} du$$

$$= \frac{2\pi}{2\omega} = \frac{\pi}{\omega}.$$

Hence the time-averaged force on the ring is

$$\boxed{\langle \vec{F} \rangle = \frac{\pi a^2 b^2}{R}\left(\mu_0 n I_0\right)^2 \frac{d\theta}{dz}\,\hat{z}}$$