# UNM Physics 262, Problem Set 2, Fall 2006

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Do all of the exercises and problems listed below. Hand in your problem set in the rolling cart hand-in box, either before class or after class, or in the box in the Physics and Astronomy main office by 5 p.m. Be sure to put your name or CPS number on your problem set as well as the course number (Physics 262). Show all your work, write clearly, indicate directions for all vectors, and be sure to include the units! Credit will be awarded for clear explanations as much, if not more so, than numerical answers. Avoid the temptation to simply write down an equation and move symbols around or plug in numbers. Explain what you are doing, draw pictures, and check your results using common sense, limits, and/or dimensional analysis.

Exercises: Young & Freedman 32.13

Problems: Young & Freedman 32.51, 33.48, 33.60, and the problem below.

Extra Credit: Young & Freedman Problems 33.52 and 33.53

**I. Deriving the Poynting vector and EM field energy using Maxwell's equations.** In class, based on general energy conservation principles, I argued that the following equation should hold for an electromagnetic field:

$$\frac{d}{dt}\int u\, d^3r + \int \vec{\mathbf{S}} \cdot \hat{\mathbf{n}}\, da = -\int \vec{\mathbf{E}} \cdot \vec{\mathbf{J}}\, d^3r. \tag{1}$$

Here $u$ is the field's energy density, $\vec{\mathbf{S}}$ is the field's energy flux (the Poynting vector), $\vec{\mathbf{E}}$ is the electric part of the field, and $\vec{\mathbf{J}}$ is the current density of moving charges.

We will follow an analysis similar to the one Poynting used to derive his famous formula in 1884. The general strategy is to apply Maxwell's equations to the right hand side of this equation to show that

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) \tag{2}$$

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\vec{\mathbf{E}} \times \vec{\mathbf{B}}. \tag{3}$$

Our tactics for implementing this strategy will be to consider a series of infinitesimal loops, surfaces, and volumes, much as we did in the derivation of plane wave solutions of Maxwell's equations. By using these tactics, we will be able to express $\vec{\mathbf{J}}$ solely in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ fields, enabling Eqs. (2) and (3) to follow with just a little extra math.

To make this problem more manageable, it has been broken into many smaller steps. This is the way most practicing physicists tackle tough research problems: they devise a general strategy, assemble a set of problem tactics that will implement that strategy, and then divide each tactic into small pieces that can be solved using straightforward methods.

To begin, let's set up a coordinate system. The coordinate system we will use will have $\hat{\mathbf{z}}$ pointing upwards, $\hat{\mathbf{y}}$ pointing to the right, and $\hat{\mathbf{x}}$ pointing out of the page. Let $dV$ be an infinitesimal cube in the positive octant of this coordinate system having sides of lengths $dx$, $dy$, and $dz$ in the $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ directions respectively.

*a*) Draw this infinitesimal cube, labeling coordinate axes and side lengths along all three dimensions.

Inside this volume $dV$, we suppose that there is an electric field, a magnetic field, and a current density, each of which can be generally and respectively expressed as

$$\begin{aligned}
\vec{\mathbf{E}} &= E_x(x,y,z,t)\hat{\mathbf{x}} + E_y(x,y,z,t)\hat{\mathbf{y}} + E_z(x,y,z,t)\hat{\mathbf{z}} \\
\vec{\mathbf{B}} &= B_x(x,y,z,t)\hat{\mathbf{x}} + B_y(x,y,z,t)\hat{\mathbf{y}} + B_z(x,y,z,t)\hat{\mathbf{z}} \\
\vec{\mathbf{J}} &= J_x(x,y,z,t)\hat{\mathbf{x}} + J_y(x,y,z,t)\hat{\mathbf{y}} + J_z(x,y,z,t)\hat{\mathbf{z}}.
\end{aligned}$$

Although $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$, and $\vec{\mathbf{J}}$ may vary within the cube $dV$, we will assume that at the origin of the cube the vector fields have the known values

$$\begin{aligned}
\vec{\mathbf{E}} &= E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}} + E_z\hat{\mathbf{z}} \\
\vec{\mathbf{B}} &= B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} \\
\vec{\mathbf{J}} &= J_x\hat{\mathbf{x}} + J_y\hat{\mathbf{y}} + J_z\hat{\mathbf{z}}
\end{aligned}$$

In our first part of the analysis of what's going on in this cube, we will consider an infinitesimal loop going around the perimeter of the **bottom** face of this cube in a *counter-clockwise* direction as viewed from above.

*b*) Draw this infinitesimal loop, labeling coordinate axes and side lengths and showing the orientation of the circulation considered.

As we move away from the origin where $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ and $\vec{\mathbf{J}}$ are known, these vector fields will change. Using a first-order Taylor expansion (a linear extrapolation approximation), these vector fields at an arbitrary point $(x,y,z)$ away from the origin will be

$$\begin{aligned}
\vec{\mathbf{E}} &= (E_x + x\partial_x E_x)\hat{\mathbf{x}} + (E_y + y\partial_y E_y)\hat{\mathbf{y}} + (E_z + z\partial_z E_z)\hat{\mathbf{z}} \\
\vec{\mathbf{B}} &= (B_x + x\partial_x B_x)\hat{\mathbf{x}} + (B_y + y\partial_y B_y)\hat{\mathbf{y}} + (B_z + z\partial_z B_z)\hat{\mathbf{z}} \\
\vec{\mathbf{J}} &= (J_x + x\partial_x J_x)\hat{\mathbf{x}} + (J_y + y\partial_y J_y)\hat{\mathbf{y}} + (J_z + z\partial_z J_z)\hat{\mathbf{z}}.
\end{aligned}$$

Here we have used the abbreviated notation $\partial_x \equiv \partial/\partial x$, *etc.*

Consider the line integral $\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}$ around the loop depicted in part *b*. Using the extrapolations above, this line integral to lowest nontrivial order (that is, ignoring terms that have a product of more than two differentials in them) is

$$\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}} = B_x dx + (B_y + \partial_x B_y dx)dy - (B_x + \partial_y B_x dy)dx - B_y dy$$
$$= (\partial_x B_y - \partial_y B_x)dxdy.$$

The flux of $\vec{\mathbf{J}}$ through the surface bounded by this loop is

$$\int \vec{\mathbf{J}} \cdot \hat{\mathbf{n}}\, da = J_z dxdy$$

and the flux of the time derivative of $\vec{\mathbf{E}}$ through this surface is

$$\int \partial_t \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, da = \partial_t E_z dxdy$$

*c*) Write down the Amperé-Maxwell law and apply it to the line and surface integrals above. Use what you find to express $J_z$ in terms of $B_x$, $B_y$, $E_z$, and their derivatives.

Now consider the line integral $\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}}$ around the same loop. By analogy, to lowest order this evaluates to

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}} = (\partial_x E_y - \partial_y E_x)dxdy.$$

The flux of $\vec{\mathbf{B}}$ through the surface bounded by this loop is

$$\int \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\, da = B_z dxdy.$$

*d*) Write down Faraday's law and apply it to the line and surface integrals above. Use what you find to express $(\partial_x E_y - \partial_y E_x)$ in terms of $\partial_t B_z$.

We've finished looking at this loop for a while. Now consider an infinitesimal loop going around the perimeter of the **left** face of this cube in a *counterclockwise* direction as viewed from the right.

*e*) Draw this infinitesimal loop, labeling coordinate axes and side lengths and showing the orientation of the circulation considered.

*f*) In analogy with the analysis done for the loop depicted in part *b*, write down expressions for $\oint \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{\ell}}$, $\int \vec{\mathbf{J}} \cdot \hat{\mathbf{n}}\, da$, $\int \partial_t \vec{\mathbf{E}} \cdot \hat{\mathbf{n}}\, da$, $\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{\ell}}$, and $\int \vec{\mathbf{B}} \cdot \hat{\mathbf{n}}\, da$ for this loop and the surface it bounds. You may find it helpful to rotate the figure you drew in part *b* and compare it to the figure you drew in part *e*.

*g*) Write down the Amperé-Maxwell law and apply it to the line and surface integrals from part *f*. Use what you find to express $J_y$ in terms of $B_x$, $B_z$, $E_y$, and their derivatives.

$h$) Write down Faraday's law and apply it to the line and surface integrals from part $f$. Use what you find to express $(\partial_x E_z - \partial_z E_x)$ in terms of $\partial_t B_y$.

$i$) Repeat the same analysis from parts $e$ through $h$ for an infinitesimal loop going around the perimeter of the **back** face of this cube in a *counterclockwise* direction as viewed from the front.

We now have solved for $\vec{\mathbf{J}}$ in terms of $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ and their derivatives. To get this in a more suitable form, we must do a little basic calculus.

$j$) Use the product rule from calculus to express $E_x \partial_t E_x$ in terms of $\partial_t(E_x E_x)$. (Hint: You should get a factor of 2 somewhere.)

$k$) Use the product rule from calculus to express $E_x \partial_y B_z$ in terms of $\partial_y(E_x B_z)$ and $B_z \partial_y E_x$.

Using the product rule as in parts $j$ and $k$, we can move derivatives around at will to suit our purposes. We are now ready to tackle the term $\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$.

$\ell$) Expand $\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$ in the components of the coordinate system we are using and plug in the expressions found for $J_x$, $J_y$, and $J_z$ found in parts $c$, $g$, and $i$. Use the product rule to express all spatial derivatives so that they act either on terms that are products of $E$ and $B$ fields or terms that are just $E$ fields. (The derivatives can still multiply other variables; we're just requiring that the objects that the derivatives act on have one of these two forms.)

$m$) Gather the terms in part $\ell$ that are spatial derivatives of $E$ fields so that you can substitute in the expressions found for $(\partial_x E_y - \partial_y E_x)$, *etc.* in parts $d$, $h$, and $i$ into this equation. Perform this substitution. Re-express all terms like $B_x \partial_t B_x$ as $\partial_t B_x B_x$ with the appropriate factor found in the product rule analysis in part $j$.

We're almost there! With the term $\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}$ in hand, we can calculate (minus) its integral over the volume $dV$, which is our objective.

$n$) Using the fundamental theorem of calculus and the results of part $m$, show that this integral can be expressed as

$$
\begin{aligned}
-\int \vec{\mathbf{E}} \cdot \vec{\mathbf{J}}\, d^3 r = -\int \vec{\mathbf{E}} \cdot \vec{\mathbf{J}}\, dxdydz \\
&= \int \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) dxdydz \\
&+ \frac{1}{\mu_0} \int (E_z B_y - E_y B_z) dydz\Big|_{x=0}^{dx} \\
&+ \frac{1}{\mu_0} \int (E_x B_z - E_z B_x) dxdz\Big|_{y=0}^{dy} \\
&+ \frac{1}{\mu_0} \int (E_x B_y - E_y B_x) dxdy\Big|_{z=0}^{dz}.
\end{aligned}
$$

The first integral demonstrates that Eq. (2) is correct and the second integral, upon careful inspection, reveals that $\vec{\mathbf{S}} = \frac{1}{\mu_0} \vec{\mathbf{E}} \times \vec{\mathbf{B}}$ as in Eq. (3). So we have proved what we set out to prove. Whew!