Similar triangles: \( \triangle BDC \cong \triangle BEF \).

\[
\Rightarrow \quad \frac{EF}{BC} = \frac{BD}{DC}
\]

\[
\frac{\Delta h}{h} = \frac{h}{R+h} \Rightarrow \Delta = \frac{h^2}{R+h}
\]

\[
\Delta = \frac{(20 \text{ m})(315 \text{ m})}{(6.36 \times 10^6 \text{ m} + 315 \text{ m})} = 9.9 \times 10^{-4} \text{ m} \approx 1 \text{ mm}
\]

b) Near the Earth we have

\[
\vec{a} = \frac{
abla \bigg( \frac{GM}{r^2} \bigg)
}{m} = -\frac{GM}{r^2} \hat{r} = \frac{-GM}{r^2} \vec{r}
\]

with \( \hat{r} = -\hat{e}_r \), so

\[
g = \frac{GM}{r^2}
\]

Differentiating for a small change in \( r \) we have

\[
dg = -\frac{2GMr^2}{r^4} dr
\]

\( \text{(Minus sign as } r \text{ increases, } g \text{ gets weaker)} \)

Labeling the separation between two vertically separated spheres \( y \), we have

\[
y = \frac{1}{2} g \Delta t^2
\]
Near the earth, this separation changes due to changing $g$:

\[ dy = \frac{1}{2}t^2 \, dt \]

Combining these equations yields

\[ dy = -\frac{g_0 k^2}{r^3} \, dr \]

Finally, we can solve for the time of the fall $t$ by approximating the acceleration as constant:

\[ h = \frac{1}{2}gt^2 \implies t = \sqrt{\frac{2h}{g}} \]

The numerical values in the problem are from the following figure:

\[ \text{dr} = 20 \text{m} \quad h = 3.15 \text{ m} \]

Using $g = g_0$ near the earth, this yields

\[ t = \sqrt{\frac{2(3.15 \text{ m})}{9.8 \text{ m/s}^2}} \approx 0.8 \text{ s} \]

Hence, for $dy$ we get

\[ dy = -\frac{g_0 R^2 \, dh}{g_0 R^2 \, dr} \, dr = -\frac{2hR^4}{r^3} \, dr \]

The value $r$ can be substituted for the distance to either sphere. Either way, $r \approx R$ so the separation becomes...
\[ dy = -2 \left( \frac{A}{R} \right) dr = -2 \left( \frac{3.15 \times 10^5}{6.44 \times 10^8} \right) (20m) \approx 1.9 \times 10^{-3} m \]

\[ dy \approx -2 \text{ mm} \]

But wait! What does this minus sign mean? Wouldn't this mean that the spheres are getting closer together? No, it doesn't, which would have been clearer if we used vectors:

\[ \ddot{y} = -g \left( \frac{1}{R^2} \right) \]

accelerates in \( \hat{r} \) direction, i.e., towards earth.

\[ \ddot{y} = \dot{y} \Rightarrow \dot{y} = -\frac{1}{2} g t^2 \quad \text{really, so} \]

\[ dy = -\frac{1}{2} g t^2 \, dt \]

Which upon substitution as before gets the sign correct.

\[ \begin{array}{c}
\text{2 metal spheres, radii } R. \\
\text{mass density } \rho = 8000 \text{ kg/m}^3 \\
M = \frac{4}{3} \pi R^3
\end{array} \]

FBD on mass 1:

\[ \vec{F}_g \quad \text{or} \quad \vec{r} \]

Newton's 2nd Law:

\[ \Sigma \vec{F} = m \vec{a} \]

\[ \vec{F}_g = m \vec{a} \]

\[ \vec{a} = \frac{\vec{F}_g}{m} = \frac{1}{M} \left( -\frac{G M M}{r^2} \hat{r} \right) \Rightarrow a = \frac{G M}{r^2} \quad \text{toward other sphere,} \]
If we approximate the acceleration as uniform towards the other sphere, we have the kinematics equation

\[ r = r_0 + \frac{1}{2} a t^2 \]

Substituting in for \( a \) from the FBD analysis and using \( \Delta r = r - r_0 \), we obtain:

\[ \frac{\Delta r}{a} = \frac{G m}{r_0^2} = \frac{G}{r_0^2} \left( \frac{a}{\pi \rho} \right)^{\frac{3}{2}} \]

\[ R^3 = \frac{2G r_0 \Delta r^2}{\pi \rho} \]

\[ R = \left( \frac{3 r_0^2 \Delta r}{2 \pi G \rho t^2} \right)^{\frac{2}{3}} \]

Substituting in numbers, using

\[ r_0 = 20 \text{ m} = 2 \times 10^{-1} \text{ m} \]

\[ \Delta r = 1 \text{ mm} = 1 \times 10^{-3} \text{ m} \]

\[ \rho = 8000 \text{ kg/m}^3 = 8 \times 10^3 \text{ kg/m}^3 \]

\[ G = 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \text{ s}^{-2} \]

\[ R = \left( \frac{3 \left( 2 \times 10^{-1} \text{ m} \right)^2 \left( 1 \times 10^{-3} \text{ m} \right)}{2 \pi \left( 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \text{ s}^{-2} \right) \left( 8 \times 10^3 \text{ kg/m}^3 \right) \left( 8 \times 10^3 \text{ kg/m}^3 \right)} \right)^{\frac{3}{2}} \]

\[ R = \left( \frac{3 \cdot 4 \cdot 10^{-2} \text{ m}^3}{2 \pi \left( 6.67 \times 10^{-11} \text{ Nm}^2 \text{ kg}^{-2} \text{ s}^{-2} \right)} \right)^{\frac{3}{2}} \]

\[ R = \left( \frac{3}{32 \left( 8 \pi \right)} \right)^{\frac{3}{2}} \times 10^7 \text{ m} \]

\[ R \approx 17.8 \text{ m} \]
Actually, putting slightly more care into the problem and considering Newton’s 3rd law, we should have set \( d > 0.5 \text{ m} \), because this is the half-charge of separation induced between the spheres due to their mutual gravitational attraction. This then gives

\[
R = \left( \frac{1}{2} \right)^{\frac{1}{3}} (17.8 \text{ m})
\]

\[
R = 14.1 \text{ m}
\]

These are huge test particles! At this size, they are even bigger than could fit at the 20 m separation between their centers. Evidently, very massive objects can serve as test particles before apparent violations of Newton’s 1st law are seen.
This problem is very tricky. The POR states that the Laws of Physics are the same in all IRFs. So to say that the quantity isn't necessarily the same is to say that its change can't impact the laws of physics.

The speed of an electron is **NOT** necessarily the same in the two IRFs. Determining its speed involves space and time measurements between events. These can vary from one IRF to another, whether we use the Galilean Transformations or the Lorentz Transformations. Since both transformations are consistent with the POR (but perhaps not with the constancy of the speed of light), the differences in speed so obtained can't impact the laws of physics.

The charge of the electron **IS** necessarily the same in the two IRFs. If it were not, then an observer in one of the IRFs could measure the electron charge to determine his or her "absolute velocity," which would violate the POR. Charge is intrinsic.

The answer to this question depends on what you take Newton's 2nd law to be. If you use $F = \frac{d\mathbf{p}}{dt} = \mathbf{ma}$, then the mass of an electron **IS** necessarily the same in both IRFs, because we have shown in class that $E = mc^2$ are the same in IRFs under the Galilean transformation. On the other hand, using Einstein's relativity, Newton's 2nd law must be modified to $F = \frac{d(p\gamma)}{dt} = \gamma \mathbf{ma}$, so the mass of an electron is **NOT** necessarily the same.

Unlike the other questions in this part, it depends on which theory of relativity is being used.
6.2 4. The kinetic energy of an electron is **not** necessarily the same in the two IRF's. It is a function of the speed, which we have already argued is not necessarily the same in the two IRF's.

5. The electric field of an electron is **not** necessarily the same in the two IRF's. To determine its value at a point, one must measure the force on a test charge. This force in turn can be measured by measuring the change in velocity that the force imparts to a particle of known mass. We know that using the Galilean transformation, acceleration is invariant in IRF's, but we also know that Maxwell's equations must be transformed instead by the Lorentz Transformation, so we must be using Einstein's relativity here. Using the Lorentz Transformation, one can show (e.g., see Ohanian's Equations 3.41 - 3.43) that acceleration, i.e., a change in velocity, is measured to be different in different IRF's.
6.06 In Alice's frame, the work-energy theorem states
\[ \Delta KE = \vec{F} \cdot \Delta \vec{x} = 0 \]

In Bob's frame, using the Galilean transformation, he has
\[ \Delta KE' = \frac{1}{2} m (v'_{\text{f}}^2 - v'_{\text{i}}^2) \]
\[ = \frac{1}{2} m \left[ (\vec{v}'_{\text{f}} - \vec{v}'_{\text{rel}}) \cdot (\vec{v}'_{\text{f}} - \vec{v}'_{\text{rel}}) - (\vec{v}'_{\text{i}} - \vec{v}'_{\text{rel}}) \cdot (\vec{v}'_{\text{i}} - \vec{v}'_{\text{rel}}) \right] \]
\[ = \frac{1}{2} m \left( v'_{\text{f}}^2 + v'_{\text{i}}^2 - 2 \vec{v}'_{\text{f}} \cdot \vec{v}'_{\text{rel}} - 2 \vec{v}'_{\text{i}} \cdot \vec{v}'_{\text{rel}} \right) \]
\[ = \Delta KE' - (\vec{p}'_{\text{f}} - \vec{p}'_{\text{i}}) \cdot \vec{v}'_{\text{rel}} \]
\[ \vec{F}' = \vec{F} \quad \text{(providing in class)} \]
\[ \Delta x' = \frac{1}{2} \left( x'_{\text{f}} - x'_{\text{i}} \right) \epsilon \]
\[ = \frac{1}{2} \left( x_{\text{f}} - \vec{v}_{\text{rel}} \cdot t_{\text{f}} - x_{\text{i}} + \vec{v}_{\text{rel}} \cdot t_{\text{i}} \right) \]
\[ = \Delta x - \vec{v}_{\text{rel}} (t_{\text{f}} - t_{\text{i}}) \]

Hence, his particular evaluation of this expression yields
\[ \Delta KE' - \vec{F}' \cdot \Delta x' = \Delta KE - \vec{F} \cdot \Delta x \]
\[ - (\vec{p}'_{\text{f}} - \vec{p}'_{\text{i}}) \cdot \vec{v}'_{\text{rel}} + \vec{F} \cdot \vec{v}'_{\text{rel}} (t_{\text{f}} - t_{\text{i}}) \]

By definition, \( \vec{F} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{p}'_{\text{f}} - \vec{p}'_{\text{i}}}{t_{\text{f}} - t_{\text{i}}} \), so the second two terms also cancel.
\[ \boxed{\Delta KE' - \vec{F}' \cdot \Delta x' = 0} \]

Work-energy theorem, valid.
Outbound Trip

\[ A \rightarrow B \rightarrow x \]

\[ \vec{V}_{\text{plane}} = \vec{V} + \vec{V} = \frac{\Delta X_{AB}}{\Delta T_{AB}} \]

\[ \Delta T_{AB} = \frac{\Delta X_{AB}}{c-V} \]

Total time for round trip is

\[ \Delta t = \Delta T_{AB} + \Delta T_{BA} = \Delta X_{AB} \left( \frac{1}{c-V} + \frac{1}{c+V} \right) \]

\[ = \frac{\Delta X_{AB}}{c} \left( \frac{c+V+c-V}{c^2-v^2} \right) \]

\[ = \frac{2 \Delta X_{AB}}{c} \left( \frac{1}{1-v^2/c^2} \right) \]

If there were no wind \( (V=0) \), then \( \Delta t = \frac{2 \Delta X_{AB}}{c} \)

so we have

\[ \Delta T_{AB}^{\text{wind}} = \left( \frac{1}{1-v^2/c^2} \right) \Delta T_{AB}^{\text{no wind}} \]

The reason that the effect of the wind doesn't average out is that the time lost on the outbound trip can't be made up for completely during the return trip. As an extreme example, suppose \( A \rightarrow B \) were separated by 2 mi and \( c = 1 \text{ mi/h} \). With no wind, the round trip takes 1 h. But with a 3 mi/h headwind, it takes 2 h just to go from \( A \rightarrow B \).
Given $\vec{V}$, $\vec{Z}$ must point partially in the $\hat{x}$-direction in the Earth's IRF; if the flight is to arrive at $C$.

$V_{\text{plane}} = \vec{C} + \vec{V}$

$|V_{\text{plane}}| = \sqrt{\vec{C}^2 - V^2}$

$\Delta t_{AC} = \frac{\Delta x_{AC}}{\sqrt{C^2 - V^2}}$

Total time for round trip is

$\Delta t = \Delta t_{AC} + \Delta t_{CA} = \frac{\Delta x_{AC}}{C} \left( \sqrt{1 - \frac{V^2}{C^2}} \right)$

If there is no wind ($V = 0$), then $\Delta t = \frac{\Delta x_{AC}}{C}$, so we have

$\Delta t_{AC} = \left( \frac{1}{\sqrt{1 - \frac{V^2}{C^2}}} \right) \Delta t_{\text{no wind}}$
Because \( 1 - \frac{v^2}{c^2} > \sqrt{1 - \frac{v^2}{c^2}} \), the ABA trip is delayed more than the ACA trip, so the Boston flight arrives first. The amount of time it beats the Calgary flight by is

\[
\Delta t = \Delta t_{\text{ABA}} - \Delta t_{\text{ACA}}
\]

\[
\Delta t = \Delta t_{\text{No. 1}} \cdot \left( \frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}} - \frac{v^2}{c^2}} \right)
\]

(iii) If we expand the \( \Delta t \) terms above with the binomial expansion, we get

\[
\frac{1}{1 - \frac{v^2}{c^2}} = (1 - \frac{v^2}{c^2})^{-1} \approx 1 + \frac{v^2}{c^2}
\]

\[
\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = (1 - \frac{v^2}{c^2})^{-\frac{1}{2}} \approx 1 + \frac{1}{2} \frac{v^2}{c^2}
\]

\[
1 - \frac{v^2}{c^2} - \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}^2} \approx 1 + \frac{v^2}{c^2} - 1 - \frac{1}{2} \frac{v^2}{c^2} = \frac{1}{2} \frac{v^2}{c^2}
\]

\[\Rightarrow \Delta t = \Delta t_{\text{No. 1}} \cdot \left( \frac{1}{2} \frac{v^2}{c^2} \right)\]

Using \( \Delta t_{\text{No. 1}} = \frac{L}{c} \) (L is round trip distance), we obtain

\[\Delta t = \frac{L}{2c} \left( \frac{v^2}{c^2} \right)\]
$6.3\ d$

A "square" oval.

\[ \vec{v}_{\text{wind}} = \frac{5280 \text{ft}}{1 \text{min}} \times \frac{1 \text{ ft}}{12 \text{ in}} \times \frac{1 \text{ in}}{100 \text{ cm}} = \frac{0.84 \text{ m/s}}{1 \text{ min}} \]

For a runner around the track, we must add the time delays caused by the \( \vec{v}_{\text{wind}} \) (if they moved against the wind, not the ground, which is unrealistic.)

\[ \Delta t = \Delta t_{\text{nom}} \left( \frac{1 - \frac{v^2}{c^2}}{1 - \frac{v^2}{c^2}} + \sqrt{1 - \frac{v^2}{c^2}} \right) \]

\[ = \frac{1}{2} (43.18) \left( 1 - \frac{0.094}{0.264} \right)^2 + \left( 1 - \frac{0.094}{0.264} \right) \frac{1}{2} \]

\[ = \frac{1}{2} (43.18) (2.014) \]

\[ \Delta t = 43.48 \text{ s} \rightarrow \boxed{MJ \ is \ 0.20 \text{ s slower}} \]

Notice that if we used the binomial approximation, we would have found the following:

\[ \Delta t \approx \Delta t_{\text{nom}} \left( 2 + \frac{v^2}{c^2} \right) \]

\[ = (43.18) \left( 1 + \frac{3}{2} \left( \frac{0.094 \text{ m/s}}{0.264 \text{ m/s}} \right)^2 \right) \]

\[ = (43.18) (1.007) = 43.48 \text{ s} \rightarrow 0.20 \text{ s slower} \]

So the approximation is valid here and saves us from having to enter so much into the calculator.
Kipketer with wind:

\[ \Delta t \approx \Delta t \text{ no wind} \left(1 + \frac{3}{4} \frac{V}{c} \right) \left[ \text{can use same factor, even though total distance traveled differs now} \right] \]

\[ \frac{\text{Kipketer}}{800 \text{ m}} = 7.912 \text{ m/s} \]

\[ V_{\text{wind}} = 2 \text{ mph} = 0.894 \text{ m/s} \]

\[ \Delta t_{\text{wind}} \leq (101.115) \left(1 + \frac{3}{4} \left(\frac{0.894 \text{ m/s}}{7.912 \text{ m/s}}\right)^2 \right) \left[ \text{Binomial approx.} \right] \]

\[ \leq (101.115)(1.010) \]

\[ = 102.085 \text{ s} \]

\[ \Delta t_{\text{wind}} - \Delta t_{\text{no wind}} = 0.35 \text{ s} \]

\[ \therefore \text{ Coe could have beaten Kipketer by 0.35 s!} \]
Problem 10.3 has vertical dimension \( h \) and horizontal dimensions \( l_1 \) and \( l_2 \). Let \( l = \max(l_1, l_2) \), as it is the largest horizontal dimension that will be the most sensitive to accelerations due to changes in the Earth's gravity.

\[
\begin{align*}
\text{In 6.1, we found that the horizontal separation between test masses decreases by an amount} \\
dx = \frac{2h}{R} \\
\text{and that the vertical separation between test masses increases by an amount} \\
dy = \frac{2h}{R} \\
\text{For test masses originally separated an amount } \Delta r \text{ that falls an amount } h_{\text{fall}}. \\
The product } h_{\text{fall}} \Delta r \text{ is maximized for } h_{\text{fall}} = 3h = 30 \text{ m, so} \\
dy = \frac{h}{R} \\
\text{To compute } dx \text{ and } dy \text{ we just need estimates for } h + l.
\end{align*}
\]

I use

\[
\begin{align*}
h &\sim (30 \text{ ft}) \left( \frac{2.54 \text{ cm}}{1 \text{ in}} \right) \left( \frac{1 \text{ m}}{100 \text{ cm}} \right) \approx 0.9144 \text{ m} \approx 100 \\
l &\sim (90 \text{ ft}) = 3h = 30 \text{ m}
\end{align*}
\]
\[ \text{dx} \sim 10\ m \left( \frac{10^{-3} \text{ m}}{6300 \text{ km/10}^3 \text{ m}} \right) = 6.3 \times 10^{-6} \text{ m} \sim 2 \times 10^{-5} \text{ m} \]

\[ \text{dy} \sim 3 \text{ dx} \]

\[ \Rightarrow \text{To precision of about } 20 \text{ mm, } \pm \frac{1}{8} \text{ an RF} \]

The time associated with this is the time of a fall from \( \frac{h}{2} \):

\[ \frac{1}{2} = \frac{1}{2} g \frac{h^2}{2} \quad t = \sqrt{\frac{h}{g}} = \sqrt{\frac{10^3 \text{ m}}{10^4 \text{ m/s}^2}} \sim 1 \text{ s} \]

(6) From 6.1, objects are good that masses to a position \( \Delta r \) over a time \( \Delta t \) if

\[ \frac{2 \Delta r}{(\Delta t)^2} = \frac{6 \text{ m}}{r_0^2} \quad M = \frac{2 \Delta r}{6 \text{ m/s}^2} \]

Using \( r = l = 30 \text{ m}, \Delta r = 20 \text{ mm}, \Delta t = 1 \text{ s}, G = 6 \times 10^{-11} \text{ N m}^2/\text{kg}^2 \)

\[ M = \frac{2(30 \text{ m})(2 \times 10^{-3} \text{ m})}{(6 \times 10^{-11} \text{ N m}^2/\text{kg}^2)(1 \text{ s})} \sim 2 \times 10^{-4} + 11 \text{ kg} \sim 2 \times 10^7 \text{ kg} \]

\[ M = 20 \text{ million kg} \quad (\text{Heavy!}) \]
To get the velocity transformation, we take the derivative w.r.t. time of how the rotation transformation acts on position coordinates.

\[ t' = t \]
\[ x' = x \cos \theta + y \sin \theta \]
\[ y' = -x \sin \theta + y \cos \theta \]
\[ z' = z \]

\[
\begin{align*}
\frac{dt'}{dt} &= 1 \\
\frac{dx'}{dt'} &= \left( \frac{dx'}{dt} \right) \frac{dt}{dt'} = \frac{dx}{dt} \cos \theta + \frac{dy}{dt} \sin \theta \\
\frac{dy'}{dt'} &= \left( \frac{dy'}{dt} \right) \frac{dt}{dt'} = -\frac{dx}{dt} \sin \theta + \frac{dy}{dt} \cos \theta \\
\frac{dz'}{dt'} &= \frac{dz'}{dt} \frac{dt}{dt'} = \frac{dz}{dt}
\end{align*}
\]

Using \( V_x = \frac{dx}{dt}, \ V_y = \frac{dy}{dt}, \ V_z = \frac{dz}{dt} \), this becomes

\[
\begin{pmatrix}
V_x' \\
V_y' \\
V_z'
\end{pmatrix} =
\begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{pmatrix}
V_x \\
V_y \\
V_z
\end{pmatrix}
\]
6.5.6  Taking one more time derivative we have
\[
\frac{dV_x}{dt} = \frac{dV_x}{dt} \frac{d\theta}{d\theta} = \frac{dV_y}{dt} \cos \theta + \frac{dV_y}{dt} \sin \theta
\]
\[
\frac{dV_y}{dt} = \frac{dV_y}{dt} \frac{d\theta}{d\theta} = -\frac{dV_x}{dt} \sin \theta + \frac{dV_x}{dt} \cos \theta
\]
\[
\frac{dV_z}{dt} = \frac{dV_z}{dt} \frac{d\theta}{d\theta} = \frac{dV_z}{dt}
\]
Using \( a_x = \frac{dV_x}{dt} \), etc., this transformation becomes
\[
\begin{bmatrix}
\alpha_x' \\
\alpha_y' \\
\alpha_z'
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\alpha_x \\
\alpha_y \\
\alpha_z
\end{bmatrix}
\]

2. The rotation transformation acts on unit vectors the same way it does on positions:
\[
\begin{bmatrix}
\hat{x}' \\
\hat{y}' \\
\hat{z}'
\end{bmatrix}
= \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix}
\]

(4) Putting it all together, Bob says that \( m\hat{a}' \) is:
\[
m\hat{a}' = m\left[ (a_x \cos \theta + a_y \sin \theta) (\cos \theta \hat{x} + \sin \theta \hat{y}) \\
+ (-a_x \sin \theta + a_y \cos \theta) (-\sin \theta \hat{x} + \cos \theta \hat{y}) + a_z \hat{z} \right]
\]
\[
= m \left[ \hat{x} \left( a_x \cos^2 \theta + a_y \sin \theta \cos \theta + a_x \cos \theta \sin \theta - a_x \cos \theta \sin \theta \right) \\
+ \hat{y} \left( a_x \cos \theta \sin \theta + a_y \sin^2 \theta - a_x \cos \theta \sin \theta + a_y \cos \theta \right) + \hat{z} a_z \right]
\]
\[
= m \left[ a_x \hat{x} + a_y \hat{y} + a_z \hat{z} \right] = \vec{m}\hat{a} \quad \therefore \quad \vec{m}\hat{a}' = \vec{m}\hat{a}
\]