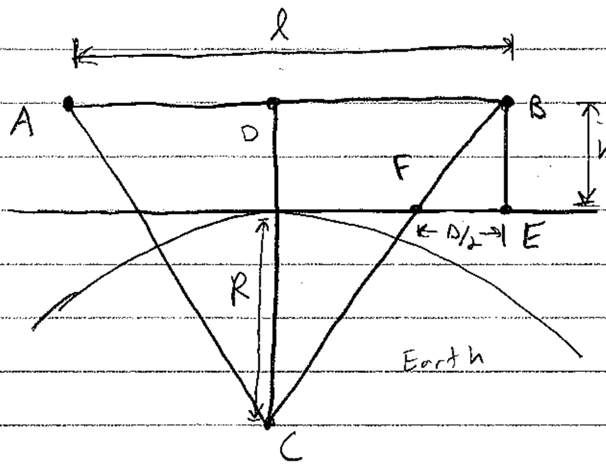
Ph262 PS 6

6.1

(a)



$l = 20\,m$
$h = 315\,m$
$R = 6.36 \times 10^6\,m$

Similar triangles: $BDC \cong BEF$.

$$\Rightarrow \frac{\overline{EF}}{\overline{BE}} = \frac{\overline{BD}}{\overline{DC}}$$

$$\frac{\Delta/2}{h} = \frac{l/2}{R+h} \Rightarrow \Delta = \frac{lh}{R+h}$$

$$\Delta = \frac{(20\,m)(315\,m)}{(6.36 \times 10^6\,m + 315\,m)} = \boxed{9.9 \times 10^{-4}\,m \sim 1\,mm}$$

(b) Near the Earth we have

$$\vec{g} = \frac{\vec{F}}{m} = -\frac{GM}{r^2}\hat{r} = \frac{GM}{R^2}\frac{R^2}{r^2}\hat{r} = \frac{-g_0 R^2}{r^2}\hat{r} \quad \text{with } \vec{g} = -g\hat{r}, \text{ so}$$

$$g = \frac{g_0 R^2}{r^2}$$

Differentiating for a small change in r we have

$$dg = \frac{-2g_0 R^2}{r^3}\,dr \qquad (\text{Minus sign: as } r \text{ increases, } g \text{ gets weaker})$$

Labeling the separation between 2 vertically separated spheres y, we have

$$y = \frac{1}{2} g t^2$$

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(6.1) (b) Near the earth, this separation changes due to a change in g:

$$dy = \frac{1}{2} t^2 \, dg$$

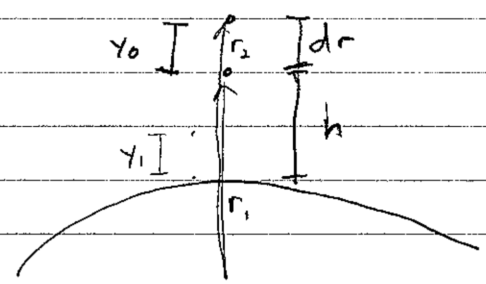
Combining these equations yields

$$dy = -\frac{g_0 R^2 t^2}{r^3} \, dr$$

Finally, we can solve for the time of the fall t by approximating the acceleration as constant:

$$h = \frac{1}{2} g t^2 \Rightarrow t = \sqrt{\frac{2h}{g}}.$$

The numerical values in the problem are from the following figure:



$dr = 20 m$
$h = 315 m$

Using $g = g_0$ near the earth, this yields

$$t = \sqrt{\frac{2(315 m)}{9.8 m/s^2}} \approx 8.02 s$$

Hence for dy we get

$$dy = -\frac{\cancel{g_0} R^2 (2h)}{\cancel{g_0} r^3} dr = -\frac{2hR^2 dr}{r^3}$$

The value r can be substituted for the distance to either sphere. Either way, $r \sim R$ so the separation becomes

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$$dy = -2\left(\frac{h}{R}\right)dr = -2\left(\frac{315\,m}{6.4\times10^{6}\,m}\right)(20m) \cong 1.9\times10^{-3}\,m$$

$$\boxed{dy \approx -2\,mm}$$

But wait! What does this minus sign mean? Wouldn't this mean that the spheres are getting closer together? No, it doesn't, which would have been clearer if we used vectors:

$$\vec{y} = -\frac{1}{2} g t^2 \hat{r}$$

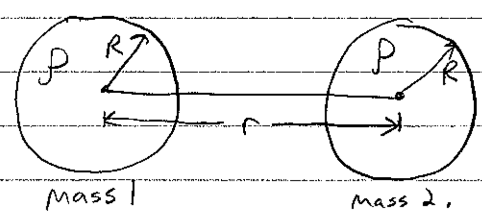
accelerates in $-\hat{r}$ direction, i.e. <u>towards</u> earth.

$$\vec{y} = y\hat{r} \Rightarrow y = -\frac{1}{2} g t^2 \text{ really, so}$$

$$dy = -\frac{1}{2} t^2 dg$$

Which upon substitution as before gets the sign correct.

(c)



2 steel spheres, radii R.
mass density $\rho = 8000\,kg/m^3$
$M = \frac{4}{3}\pi R^3$

FBD on mass 1: $\bullet\!\!\xrightarrow{\vec{F}_g}$ $\quad \overleftarrow{\ \hat{r}\ }$

Newton's 2nd Law: $\sum \vec{F} = m\vec{a}$

$$\vec{F}_g = m\vec{a}$$

$$\vec{a} = \frac{\vec{F}_g}{m} = \frac{1}{m}\left(\frac{-GMm}{r^2}\hat{r}\right) \Rightarrow a = \frac{GM}{r^2} \text{ toward other sphere.}$$

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6.1 c If we approximate the acceleration as uniform towards the other sphere, we have the kinematics equation

$$r = r_0 + \frac{1}{2}at^2$$

Substituting in for a from the FBD analysis and using $\Delta r \equiv r - r_0$ we obtain:

$$\frac{2\Delta r}{t^2} = a = \frac{Gm}{r_0^2} = \frac{G}{r_0^2}\left(\frac{4}{3}\pi R^3 \rho\right)$$

$$R^3 = \frac{2\Delta r \cdot r_0^2\, 3}{{}_2 4\pi\, t^2\, G\rho}$$

$$R = \left(\frac{3 r_0^2 \Delta r}{2\pi G \rho t^2}\right)^{1/3}$$

Substituting in numbers, using

$r_0 = 20\,m = 2\times10^1\,m$

$\Delta r = 1\,mm = 1\times10^{-3}\,m$

$\rho = 8000\,kg/m^3 = 8\times10^3\,kg/m^3$

$G = 6.67\times10^{-11}\,\frac{Nm^2}{kg^2}$

$1\,N = \frac{kg\,m}{s^2}$

$$R = \left(\frac{3\,(2\times10^1 m)^2\,(1\times10^{-3}\cancel{m})}{2\pi\,(6.67\times10^{-11}\,\cancel{N}\cancel{m^2}/\cancel{kg^2})(8\times10^3\,\cancel{kg}/\cancel{m^3})(8\,\cancel{s})^2\,\frac{\cancel{kg}\,\cancel{m}}{\cancel{N}\,\cancel{s^2}}}\right)^{1/3}$$

$$R = \left(\frac{3\cdot\cancel{4}^{\,\cancel{2}}\cdot10^{\cancel{2-3}\;-1}\;\;m^3}{\cancel{2}\pi\,(6.67)(8)\underbrace{(\cancel{64})}_{32}\,(10^{\cancel{-11+3}\;-8})}\right)^{1/3}$$

$$= \left(\frac{3}{32(8\pi)(6.67)}\times10^7\right)^{1/3}\ m$$

$R = 17.8\,m$

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6.1 (c) Actually, putting slightly more care into the problem and considering Newton's 3rd law, we should have set $\Delta r = 0.5\,mm$, because this is the half-change of separation induced between the spheres due to their mutual gravitational attraction. This then gives

$$R = \left(\frac{1}{2}\right)^{1/3}(17.8\,m)$$

$$\boxed{R = 14.1\,m}$$

These are huge test particles! At this size, they are even bigger than could fit at the 20 m separation between their centers. Evidently, very massive objects can serve as test particles before apparent violations of Newton's 1st law are seen.

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(6.2) (a) This problem is very tricky. The POR states that the Laws of Physics are the same in all IRFs. So to say that the quantity isn't necessarily the same is to say that its change won't impact the laws of physics.

(1) The speed of an electron is $\boxed{\text{NOT}}$ necessarily the same in the two IRFs. Determining its speed involves space and time measurements between events. These can vary from one IRF to another, whether we use the Galilean Transformation or the Lorentz Transformation. Since both transformations are consistent with the POR (but perhaps not w/ the constancy of the speed of light), the differences in speed so obtained cannot impact the laws of physics.

(2) The charge of the electron $\boxed{\text{IS}}$ necessarily the same in the two IRFs. If it were not, then an observer in one of the IRFs could measure the electron charge to determine his or her "absolute velocity," which would violate the POR. Charge is intrinsic.

(3) The answer to this question depends on what you take Newton's 2nd law to be. If you use $\vec{F} = d\vec{p}/dt = m\vec{a}$, then the mass of an electron $\boxed{\text{IS}}$ necessarily the same in both IRFs, because we have shown in class that $\vec{F}$ and $\vec{a}$ are the same in IRFs under the Galilean transformation. On the other hand, using Einstein's relativity, Newton's 2nd law must be modified to $\vec{F} = d(\gamma\vec{p})/dt = \gamma m\vec{a}$, so the mass of an electron is $\boxed{\text{NOT}}$ necessarily the same. Unlike the other questions in this part, it depends on which theory of relativity is being used.

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6.2 (a) ④ The kinetic energy of an electron is $\boxed{\text{NOT}}$ necessarily the same in the two IRFs. It is a function of the speed, which we have already argued is not necessarily the same in the two IRFs.

⑤ The electric field of an electron is $\boxed{\text{NOT}}$ necessarily the same in the two IRFs. To determine its value at a point, one must measure the force on a test charge. This force in turn can be measured by measuring the change in velocity that the force imparts to a particle of known mass. We know that using the Galilean transformation, acceleration is invariant in IRFs, but we also know that Maxwell's equations must be transformed instead by the Lorentz Transformation, so we must be using Einstein's relativity here. Using the Lorentz transformation, one can show (e.g., see Ohanian's equations (2.41 - 2.43)) that acceleration, i.e. a change in velocity, is measured to be different in different IRFs.

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6.2 (a) In Alice's frame, the work-energy theorem states

$$\Delta KE - \vec{F} \cdot \Delta \vec{x} = 0$$

In Bob's frame, using the Galilean transformation he has

$$\Delta KE' = \tfrac{1}{2} m \left(V_f'^2 - V_i'^2 \right)$$

$$= \tfrac{1}{2} m \left[(\vec{V}_f - \vec{V}_{rel}) \cdot (\vec{V}_f - \vec{V}_{rel}) - (\vec{V}_i - \vec{V}_{rel}) \cdot (\vec{V}_i - \vec{V}_{rel}) \right]$$

$$= \tfrac{1}{2} m \left(V_f^2 + V_i^2 - 2 \vec{V}_f \cdot \vec{V}_{rel} - 2 \vec{V}_i \cdot \vec{V}_{rel} \right)$$

$$= \Delta KE - (\vec{p}_f - \vec{p}_i) \cdot \vec{V}_{rel}$$

$$\vec{F}' = \vec{F} \qquad \text{(proved in class)}$$

$$\Delta \vec{x}' = \vec{x}'_f - \vec{x}'_i$$

$$= \vec{x}_f - \vec{V}_{rel} t_f - \vec{x}_i + \vec{V}_{rel} t_i$$

$$= \Delta \vec{x} - \vec{V}_{rel} (t_f - t_i)$$

Be careful! These two times are different! This is the key

Hence his particular evaluation of this expression yields

$$\Delta KE' - \vec{F}' \cdot \Delta \vec{x}' = \underbrace{\Delta KE - \vec{F} \cdot \Delta \vec{x}}_{0 \text{ by Alice's work-energy thm.}}$$

$$- (\vec{p}_f - \vec{p}_i) \cdot \vec{V}_{rel} + \vec{F} \cdot \vec{V}_{rel} (t_f - t_i)$$

But by definition, $\vec{F} = \frac{\Delta \vec{p}}{\Delta t} = \frac{\vec{p}_f - \vec{p}_i}{t_f - t_i}$, so the second two terms also cancel:

$$\boxed{\Delta KE' - \vec{F}' \cdot \Delta \vec{x}' = 0}$$

Work-energy thm. valid ✓

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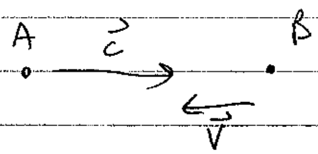
6.3 (a) (i)

Outbound Trip

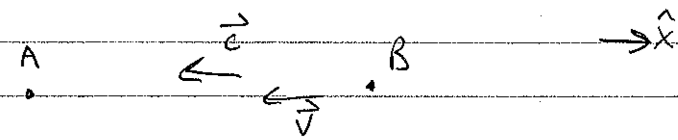


In Earth's IRF,

$$\vec{V}_{plane} = \vec{c} + \vec{V} = \frac{\Delta x_{AB}}{\Delta t_{AB}}\hat{x}$$

$$\Delta t_{AB} = \frac{\Delta x_{AB}}{c-V}$$

Return Trip



In Earth's IRF,

$$\vec{V}_{plane} = \vec{c} + \vec{V} = \frac{\Delta x_{BA}}{\Delta t_{BA}}\hat{x}$$

$$\Delta t_{BA} = \frac{\Delta x_{AB}}{c+V}$$

Total time for round trip is

$$\Delta t = \Delta t_{AB} + \Delta t_{BA} = \Delta x_{AB}\left(\frac{1}{c-V} + \frac{1}{c+V}\right)$$

$$= \Delta x_{AB}\left(\frac{c+\cancel{V}+c-\cancel{V}}{c^2-V^2}\right)$$

$$= \frac{2\Delta x_{AB}}{c}\left(\frac{1}{1-V^2/c^2}\right)$$

If there were no wind ($V=0$), then $\Delta t = \frac{2\Delta x_{AB}}{c}$, so we have

$$\boxed{\Delta t_{ABA}^{wind} = \left(\frac{1}{1-V^2/c^2}\right)\Delta t_{ABA}^{No\ wind}}$$

(ii) The reason that the effect of the wind doesn't average out is that the time lost on the outbound trip can't be made up for completely during the return trip. As an extreme example, suppose A + B were separated by 2 mi and c = 4 mi/h. With no wind, the round trip takes 1 h. But with a 3 mi/h headwind, it takes 2 h just to go from A to B!

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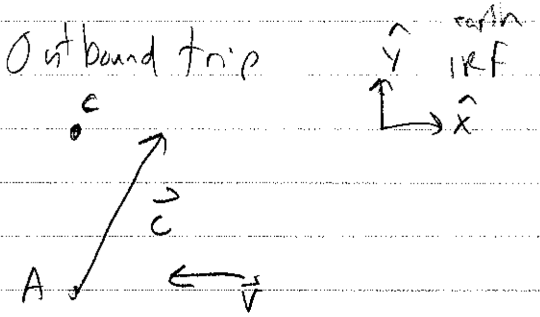
63 b) Outbound trip



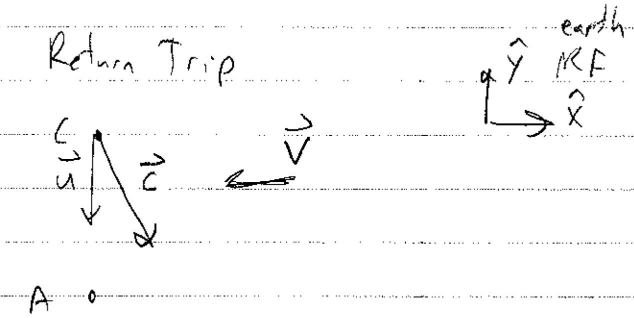
Given $\vec{v}$, $\vec{c}$ must point partially in the $\hat{x}$-direction in the Earth's IRF if the flight is to arrive at C

$\vec{V}_{plane} = \vec{c} + \vec{v}$

$|V_{plane}| = \sqrt{c^2 - v^2}$

$\Delta t_{AC} = \frac{\Delta x_{AC}}{\sqrt{c^2-v^2}}$

Return Trip



Given $\vec{v}$, $\vec{c}$ must point partially in the $\hat{x}$-direction in the Earth's IRF if the flight is to arrive at A.

$\vec{V}_{plane} = \vec{c} + \vec{v}$

$|V_{plane}| = \sqrt{c^2 - v^2}$

$\Delta t_{CA} = \frac{\Delta x_{AC}}{\sqrt{c^2-v^2}}$

Total time for round trip is

$$\Delta t = \Delta t_{AC} + \Delta t_{CA} = \frac{2\Delta x_{AC}}{c}\left(\frac{1}{\sqrt{1-v^2/c^2}}\right)$$

If there is no wind ($v=0$), then $\Delta t = \frac{2\Delta x_{AC}}{c}$, so we have

$$\boxed{\Delta t_{ACA}^{wind} = \left(\frac{1}{\sqrt{1-v^2/c^2}}\right)\Delta t_{ACA}^{no\ wind}}$$

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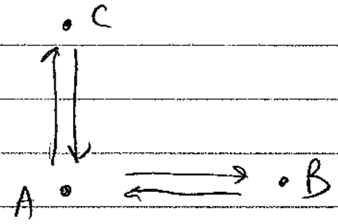
6.3 (c) (i)



Because $\frac{1}{1-v^2/c^2} > \frac{1}{\sqrt{1-v^2/c^2}}$, the ABA trip is delayed more than the ACA trip, so the Boston flight arrives home first. The amount of time it beats the Calgary flight by is

$$\Delta t = \Delta t_{ABA} - \Delta t_{ACA}$$

$$\boxed{\Delta t = \Delta t^{\text{No wind}} \left(\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}} \right)}$$

(ii) If we expand the 2 terms above with the binomial expansion, we get

$$\frac{1}{1-v^2/c^2} = \left(1-\frac{v^2}{c^2}\right)^{-1} \approx 1+\frac{v^2}{c^2}$$

$$\frac{1}{\sqrt{1-v^2/c^2}} = \left(1-\frac{v^2}{c^2}\right)^{-1/2} \approx 1+\frac{1}{2}\frac{v^2}{c^2}$$

$$\frac{1}{1-v^2/c^2} - \frac{1}{\sqrt{1-v^2/c^2}} \approx 1+\frac{v^2}{c^2} - 1 - \frac{1}{2}\frac{v^2}{c^2} = \frac{1}{2}\frac{v^2}{c^2}$$

$$\Rightarrow \Delta t \approx \Delta t^{\text{No wind}} \left(\frac{1}{2}\frac{v^2}{c^2}\right)$$

Using $\Delta t^{\text{No wind}} = L/c$ (L is round trip distance), we obtain

$$\boxed{\Delta t \cong \frac{L}{2c}\left(\frac{v^2}{c^2}\right)}$$

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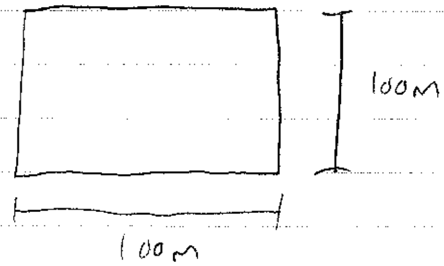
6.3 d



A "square" oval.

$$|\vec{c}| = \frac{400\,m}{43.18\,s} \approx 9.264\ m/s$$

$$\longleftarrow \vec{V}_{wind} = \frac{2\,mi}{h}\cdot\frac{h}{3600\,s}\cdot\frac{5280\,ft}{mi}\cdot\frac{12\,in}{ft}\cdot\frac{2.54\,cm}{in}\cdot\frac{m}{100\,cm} = 0.894\ m/s$$

For a runner around the track, we must add the time delays caused by the $\vec{V}$wind (if they moved against the wind, not the ground, which is unrealistic.)

$$\Delta t = \Delta t^{No\ wind}\left(\frac{1}{1-v^2/c^2} + \frac{1}{\sqrt{1-v^2/c^2}}\right)$$

$$= \frac{1}{2}(43.18\,s)\left(\frac{1}{1-\left(\frac{0.894}{9.264}\right)^2} + \frac{1}{\left(1-\left(\frac{0.894}{9.264}\right)^2\right)^{1/2}}\right)$$

$$= \frac{1}{2}(43.18\,s)(2.014)$$

$$\Delta t = 43.48\,s \quad \rightarrow \boxed{\text{MJ is } 0.20\text{ s slower}}$$

Notice that if we used the binomial approximated formula, we would have found the following:

$$\Delta t \cong \Delta t^{No\ wind}\left(2 + \frac{3}{2}\frac{v^2}{c^2}\right)$$

$$\cong (43.18\,s)\left(1 + \frac{3}{4}\left(\frac{0.894\,m/s}{9.264\,m/s}\right)^2\right)$$

$$= (43.18\,s)(1.007) = 43.48\,s \Rightarrow \boxed{0.20\text{ s slower}}$$

So the approximation is valid here and saves us from having to enter so much into the calculator!

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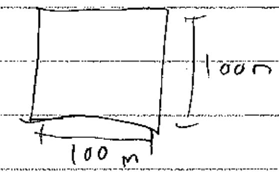
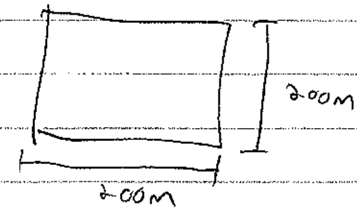
6.3 (d)



2 times around equivalent to



Kipketer: 1:41.11 $= 101.11\,s$

Coe : 1:41.73 $= 101.73\,s$

Kipketer with wind:

$$\Delta t \approx \Delta t^{\text{No wind}} \left(1 + \frac{3}{4}\frac{v^2}{c^2}\right)$$

← can use same factor, even though total distance travelled differs now

$$C_{\text{kipketer}} = \frac{800\,m}{101.11\,s} = 7.912\ m/s$$

$$V_{\text{wind}} = 2\ mph = 0.894\ m/s$$

$$\Delta t^{\text{wind}}_{\text{kipketer}} \cong (101.11\,s)\left(1 + \frac{3}{4}\left(\frac{0.894\,m/s}{7.912\,m/s}\right)^2\right)$$ [Binomial appx]

$$\cong (101.11\,s)(1.010)$$

$$= 102.08\ s$$

$$\Delta t^{\text{wind}}_{\text{Kip}} - \Delta t_{\text{Coe}} = 0.35\ s$$

∴ $\boxed{\text{Coe could have beaten Kipketer by } 0.35\ s\,!}$

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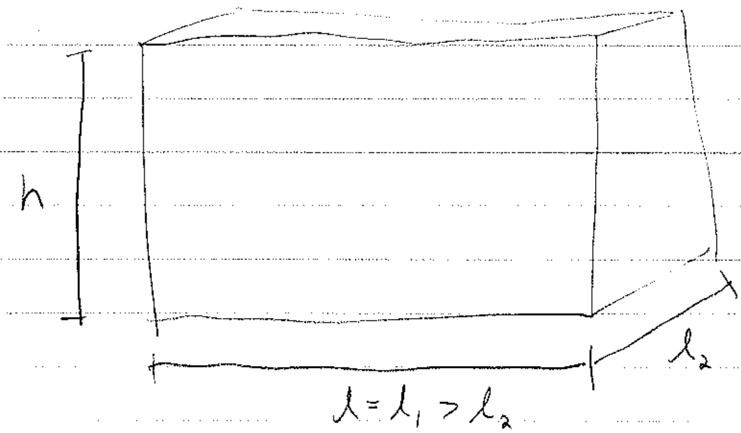
6.4 Regener 103 has vertical dimension h and horizontal dimensions l_1 and l_2. Let $l = \max(l_1, l_2)$, as it is the largest horizontal dimension that will be the most sensitive to accelerations due to changes in the Earth's gravity.



In 6.1 (a) we found that the horizontal separation between test masses decreases by an amount

$$dx = l\frac{h}{R} \qquad (R = \text{earth radius})$$

and that the vertical separation between test masses increases an amount

$$dy = \frac{2 h_{fall}\, dr}{R}$$

for test masses originally separated an amount dr that falls an amount h_{fall}. The product $h_{fall}\, dr$ is maximized for $h_{fall} = dr = h/2$, so

$$dy = h\frac{h}{R}$$

To compute dx and dy we just need estimates for h & l. I use

$$h \sim (30\text{ ft})\left(\frac{12\text{ in}}{\text{ft}}\,\frac{2.54\text{ cm}}{\text{in}}\,\frac{\text{m}}{100\text{ cm}}\right) \cong \overset{100}{\cancel{36\times 3}} \times 10^{\cancel{-2}}\text{ m} \sim 10\text{ m}$$
$$l \sim (90\text{ ft}) = 3h = 30\text{ m}$$

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6.4 So we get

$$dx \sim 10m \left(\frac{10\,m}{6300\,km} \cdot \frac{km}{10^3\,m} \right) = \frac{10 \times 10^1\,m^2}{6.3 \times 10^6\,m} \sim 2 \times 10^{1-6} = 2 \times 10^{-5}\,m$$

$$dy \sim 3\,dx$$

$\Rightarrow$ To precision of about $\boxed{20\,\mu m}$, it's an IRF

The time associated with this is the time of a fall from $h/2$:

$$\frac{h}{2} = \frac{1}{2} g t^2 \qquad t = \sqrt{\frac{h}{g}} = \sqrt{\frac{10m}{10m/s^2}} \sim \boxed{1\,s}$$

(b) From 6.1 (c), objects are good test masses to a precision Δr over a time Δt if

$$\frac{2\Delta r}{(\Delta t)^2} = \frac{G\,m}{r_0^2}, \qquad M = \frac{2 r \Delta r}{G (\Delta t)^2}$$

Using $r \sim l = 30\,m$, $\Delta r \sim 20\,\mu m$, $\Delta t \sim 1\,s$, $G = 6 \times 10^{-11}\,Nm^2/kg^2$

$$M = \frac{2(30m)(2 \times 10^{-5}m)}{(6 \times 10^{-11} \frac{Nm^2}{kg^2})(1s)^2} \sim 2 \times 10^{-4+11}\,kg \sim 2 \times 10^7\,kg$$

$\boxed{M = 20 \text{ million kg}}$ (Heavy!)

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6.5 (a) To get the velocity transformation, we take the derivative wrt. time of how the rotation transformation acts on position coordinates,

$$t' = t$$
$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\frac{dt'}{dt} = 1$$

$$\frac{dx'}{dt'} = \frac{dx'}{dt}\cancel{\frac{dt}{dt'}}^{\,1} = \frac{dx}{dt}\cos\theta + \frac{dy}{dt}\sin\theta$$

$$\frac{dy'}{dt'} = \frac{dy'}{dt}\cancel{\frac{dt}{dt'}}^{\,1} = -\frac{dx}{dt}\sin\theta + \frac{dy}{dt}\cos\theta$$

chain rule

$$\frac{dz'}{dt'} = \frac{dz'}{dt}\cancel{\frac{dt}{dt'}}^{\,1} = \frac{dz'}{dt}$$

Using $V_x' = \frac{dx'}{dt'}$, $V_y' = \frac{dy'}{dt'}$, $V_z' = \frac{dz'}{dt'}$ and

$V_x = \frac{dx}{dt}$, $V_y = \frac{dy}{dt}$, $V_z = \frac{dz}{dt}$, this becomes

$$\boxed{\begin{bmatrix} V_x' \\ V_y' \\ V_z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}}$$

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6.5 (b) Taking one more time derivative we have

$$\frac{dv_x'}{dt'} = \frac{dv_x'}{dt}\cancel{\frac{dt}{dt'}}^{1} = \frac{dv_x}{dt}\cos\theta + \frac{dv_y}{dt}\sin\theta$$

$$\frac{dv_y'}{dt'} = \frac{dv_y'}{dt}\cancel{\frac{dt}{dt'}}^{1} = -\frac{dv_x}{dt}\sin\theta + \frac{dv_y}{dt}\cos\theta$$

$$\frac{dv_z'}{dt'} = \frac{dv_z'}{dt}\cancel{\frac{dt}{dt'}}^{1} = \frac{dv_z}{dt}$$

Using $a_x' = \frac{dv_x'}{dt'}$, etc, this transformation becomes

$$\boxed{\begin{bmatrix} a_x' \\ a_y' \\ a_z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}}$$

(c) The rotation transformation acts on unit vectors the same way it does on positions:

$$\begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}' \\ \hat{y}' \\ \hat{z}' \end{bmatrix}$$

(d) Putting it all together, Bob says that $m\vec{a}'$ is:

$$m\vec{a}' = m\Big[(a_x\cos\theta + a_y\sin\theta)(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) + (-a_x\sin\theta + a_y\cos\theta)(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) + a_z\hat{z}\Big]$$

$$= m\Big[\hat{x}(a_x\cos^2\theta + \cancel{a_y\cos\theta\sin\theta} + a_x\sin^2\theta - \cancel{a_y\cos\theta\sin\theta}) + \hat{y}(\cancel{a_x\cos\theta\sin\theta} + a_y\sin^2\theta - \cancel{a_x\cos\theta\sin\theta} + a_y\cos^2\theta) + \hat{z}\,a_z\Big]$$

$$= m\big[a_x\hat{x} + a_y\hat{y} + a_z\hat{z}\big] = m\vec{a} \quad \therefore \boxed{m\vec{a}' = m\vec{a}}$$