more errors for Qubits
Need to understand

$\boxed{\text{Ph 452}}$ ~~Lecture 10~~

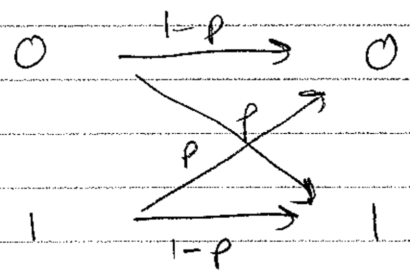
$\boxed{\text{Quantum Error Correction}}$

Noisy Channel

Hello? Hello? Hello?

Redundancy helps!

Binary symmetric channel:



$P = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$ Pgate

p is unknown!
(otherwise could apply P^{-1})

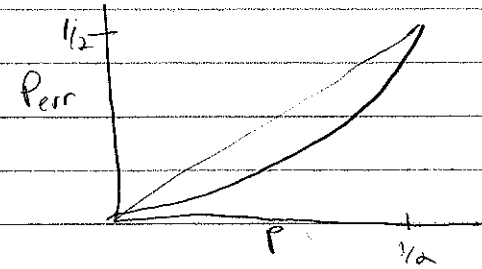
3-bit repetition code

$0 \to 000$
$1 \to 111$

Majority vote $\to$ $P_{err} = \binom{3}{3} p^3 + \binom{3}{2} p^2(1-p)$

$= p^3 + 3p^2 - 3p^3$

$= 3p^2 - 2p^3$



Improvement if $p < 1/2$!

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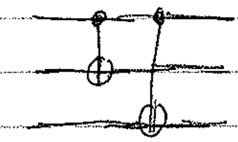
Reversible encoding:

$$0\,0\,0 \to 0\,0\,0$$
$$1\,0\,0 \to 1\,1\,1$$



What about pbits?

Encoding:

$$\vec{p} \otimes \vec{0} \otimes \vec{0} \mapsto \vec{p} \otimes \vec{p} \otimes \vec{p}$$

Impossible!

No-cloning Thm: There is no pgate that acts as $\vec{p} \otimes \vec{0} \mapsto \vec{p} \otimes \vec{p}$ for all $\vec{p} := \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$

Proof: Spse there was such a P.

Then $P(\vec{0} \otimes \vec{0}) = \vec{0} \otimes \vec{0}$

$$P(\vec{1} \otimes \vec{1}) = \vec{1} \otimes \vec{1}$$

By linearity, $P\left(\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \otimes \vec{0}\right) = \frac{1}{2} P(\vec{0} \otimes \vec{0}) + \frac{1}{2} P(\vec{1} \otimes 0)$

$$= \frac{1}{2}\left(\vec{0} \otimes \vec{0} + \vec{1} \otimes \vec{1}\right)$$

$$= \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{pmatrix}$$

But $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \otimes \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$! $\square$

→ Same for (qu)gates: No U can map $|\psi\rangle|0\rangle \mapsto |\psi\rangle|\psi\rangle \ \forall |\psi\rangle$.

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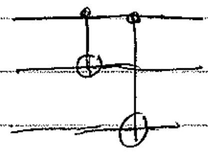
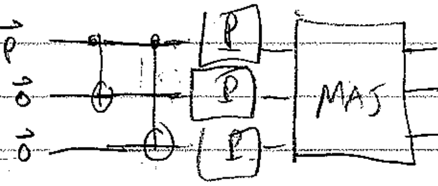
Solution: Encode pbits with same circuit used for bits!

$\vec{p} = \begin{pmatrix} p_0 \\ p_1 \end{pmatrix}$  $\begin{pmatrix} p_0 \\ 0 \\ \vdots \\ 0 \\ p_1 \end{pmatrix} = p_0\, \vec{0} \otimes \vec{0} \otimes \vec{0} + p_1\, \vec{1} \otimes \vec{1} \otimes \vec{1}$

Full encode - error - correct circuit



With a slight abuse of notation, I'll use dirac notation

$$MAJ = |000\rangle\langle 000| + |000\rangle\langle 001| + |000\rangle\langle 010| + |000\rangle\langle 100|$$
$$+ |111\rangle\langle 111| + |111\rangle\langle 110| + |111\rangle\langle 101| + |111\rangle\langle 011|$$

Not a pgate : both 001, 000 map to 000, e.g.

Not a classical measurement: MAJ is not diagonal.

MAJ is a combination of pgates + measurements

Method 1: Measure 3 bits, then correct

$\Pi_0 = |000\rangle\langle 000|$ $\qquad \Pi_3 = |011\rangle\langle 011|$

$\Pi_1 = |001\rangle\langle 001|$ $\qquad \vdots$

$\Pi_2 = |010\rangle\langle 010|$ $\qquad \Pi_7 = |111\rangle\langle 111|$

If outcome 0, apply $P_0 = I \otimes I \otimes I$
" " 1, " $P_1 = I \otimes I \otimes X$
" " 2, " $I \otimes X \otimes I$
" " 3, " $X \otimes I \otimes I$
$\vdots$ $\vdots$ $\vdots$ "
" " 7 $P_7 = I \otimes I \otimes I$

$$\boxed{\vec{p}' \mapsto P_k \Pi_k \vec{p}' \quad \text{w/ prob} \ \sum_i (\Pi_k \vec{p})_i}$$

Ph 452 Lecture 10

Method 1 fails on pbits:

E.g., suppose $p=0$ so $P = I$. Then

$$\vec{p}' := p_0|000\rangle + p_1|111\rangle \mapsto \begin{cases} |000\rangle & \text{probability } p_0 \\ |111\rangle & \quad '' \quad p_1 \end{cases}$$

But neither $|000\rangle$ nor $|111\rangle$ are $\vec{p}$!

"Measurement collapses $\vec{p}$ into one of two states, neither of which is $\vec{p}$"

Method 2: Measure the error, not the data

e.g., if in state $\underbrace{0\overbrace{1\,|\,0}^{\text{parity}}}_{\text{parity}}$ 2 parities determine error!

$$\Pi_0 = |000\rangle\langle 000| + |001\rangle\langle 001| + |110\rangle\langle 110| + |111\rangle\langle 111|$$
$$\Pi_1 = I - \Pi_0$$

$$\Pi_0' = |000\rangle\langle 000| + |100\rangle\langle 100| + |011\rangle\langle 011| + |111\rangle\langle 111|$$
$$\Pi_1' = I - \Pi_0'$$

If outcome 00, apply $P_0 = I \otimes I \otimes I$
" " 01, " $P_1 = I \otimes I \otimes X$
" " 10, " $P_2 = X \otimes I \otimes I$
" 11, " $P_3 = I \otimes X \otimes I$

$\vec{p}' \mapsto P_{ij}\, \Pi_i'\, \Pi_j\, \vec{p}'$ w/prob. $\sum_k (\Pi_i' \Pi_j \vec{p}')_k$

$\boxed{\text{Ph 452}}$ Lecture 18

Method 2 works on pbits!

$$\vec{p} \otimes \vec{0} \otimes \vec{0} = p_0 \vec{0} \otimes \vec{0} \otimes \vec{0} + p_1 \vec{1} \otimes \vec{1} \otimes \vec{1}$$

$$\xmapsto{\mathcal{P}^{\otimes 3}} p_0 (1-p)^3 |000\rangle$$
$$+ p_0 (1-p)^2 p \left(|001\rangle + |010\rangle + |100\rangle\right)$$
$$+ p_0 (1-p) p^2 \left(|011\rangle + |101\rangle + |110\rangle\right)$$
$$+ p_0 p^3 |111\rangle$$
$$+ p_1 (1-p)^3 |111\rangle$$
$$+ p_1 (1-p)^2 p \left(|110\rangle + |101\rangle + |011\rangle\right)$$
$$+ p_1 (1-p) p^2 \left(|100\rangle + |010\rangle + |001\rangle\right)$$
$$+ p_1 p^3 |000\rangle$$

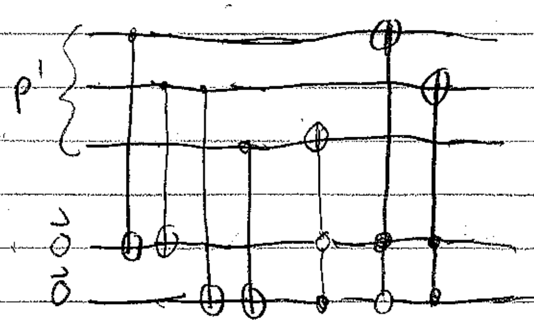
To do this coherently, need extra bits to hold the syndrome.

$$\xmapsto{R_{ij}\Pi_i\Pi_j} \begin{cases} (p_0|000\rangle + p_1|111\rangle) & \text{prob } (1-p)^3 + 3(1-p)^2 p = 1 - p_{err} \\ (p_1|000\rangle + p_0|111\rangle) & \text{prob } 3(1-p)p^2 + p^3 = p_{err} \end{cases}$$

vs. no encoding:

$$p_0|0\rangle + p_1|1\rangle \xmapsto{\mathcal{P}} (p_0|0\rangle + p_1|1\rangle)(1-p) + (p_1|0\rangle + p_0|1\rangle)p$$

Can also do method 2 w/o measurement:



$$(p_0|000\rangle + p_1|111\rangle)(1 - p_{err}) + (p_1|000\rangle + p_0|111\rangle)\, p_{err}$$

control on "0", not "1"

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This is as good a time as any to introduce...

Observables

Measurements of physical quantities have <u>units</u>

E.g. $\Pi_0 = |0\rangle\langle 0|$, $\Pi_1 = |1\rangle\langle 1|$
$f(0) = +\hbar/2$ $\quad f(1) = -\hbar/2$

Measures electron spin (units = angular momentum)

$A = \sum_i f(i)\, \Pi_i$ is an <u>observable</u> associated with $\{\Pi_i\}$ taking distinct values $\{f(i)\}$.

Abstractly, <u>any</u> $\{f(i)\}$ defines an observable, even w/o units:

E.g. $f(0) = +1$, $f(1) = -1$

$$A = \sum (+1)\,|0\rangle\langle 0| + (-1)\,|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = Z \; !$$

Similarly,
$$\left.\begin{aligned} X &= \sum |+\rangle\langle +| - |-\rangle\langle -| \\ Y &= \sum |+i\rangle\langle +i| - |-i\rangle\langle -i| \end{aligned}\right]$$
Not classical observables because projectors involved are not diagonal matrices!

Expected value:

$$\langle A \rangle = \sum p_i\, f(i)$$
$$= \sum \langle \psi | \Pi_i | \psi \rangle f(i)$$
$$= \langle \psi | \sum \Pi_i f(i) | \psi \rangle$$
$$= \langle \psi | A | \psi \rangle$$

Ph 452 Lecture 10

Example: Parity observable for 2 qubits

$\frac{1}{2}(I+Z)$
$\frac{1}{2}(I-Z)$

$$\Pi_0 = |00\rangle\langle 00| + |11\rangle\langle 11|$$
$$\Pi_1 = |01\rangle\langle 01| + |10\rangle\langle 10|$$

$$Z\otimes Z = \left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)\otimes\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right)$$
$$= \left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) - \left(|01\rangle\langle 01| + |10\rangle\langle 10|\right)$$

~~Got this far~~ $\rightarrow$ An observable for parity!

Back to 3-pbit repetition code

MAJ: ① Measure $Z\otimes Z\otimes I$, ± 1
$I\otimes Z\otimes Z$. ± 1 ← "syndrome for the error"

Shorthand: ZZI, IZZ (Don't confuse with matrix mult.)

② Apply correction ("recovery") XII, IXI, IIX as appropriate.

ZZI, IZZ & their products are called the stabilizer of the code

$$ZZI\left(p_0|000\rangle + p_1|111\rangle\right) = \left(p_0|000\rangle + p_1|111\rangle\right)$$
$$IZZ\left(\qquad\right) = \left(\quad " \quad\right)$$
$$ZIZ\left(\qquad\right) = \left(\quad " \quad\right)$$
$$III\left(\quad " \quad\right) = \left(\quad " \quad\right)$$

Next time: Encoding & correcting errors in qubits