$\boxed{\text{Ph 452}}$ Lecture 12

[Q] Quick Review

- Binary symmetric channel: Apply X w/prob p_{err} (Generally, Reversible channel: apply permutation P_i w/prob p_i — Doubly stochastic matrix by Birkhoff's Thm)
- 3-bit repetition code : $p_{err} \to O(p_{err}^2)$
- No-cloning thm : Linearity
- Reversible encoding, syndrome extraction, recovery circuits (Decoding = encoding^{-1})
- Observables : values for outcomes

Example: parity observable:

$$\Pi_0 = |00\rangle\langle 00| + |11\rangle\langle 11| \qquad f(0) = +1$$

$$\Pi_1 = |01\rangle\langle 01| + |10\rangle\langle 10| \qquad f(1) = -1$$

Real values: "Physical observable"

$$(+1)\Pi_0 + (-1)\Pi_1 = Z \otimes Z$$

Diagonal op : "Classical observable" ($\Pi = \Pi \circ I$)

Hermitian op : "Quantum observable" ($\Pi = \Pi^\dagger$)

Observables for Error correction

$$\left.\begin{aligned} S_1 &= Z \otimes Z \otimes I &= \pm 1 \\ S_2 &= I \otimes Z \otimes Z &= \pm 1 \end{aligned}\right\} \text{"Syndrome" for error}$$

Syndrome S_1	Syndrome S_2	Recovery	Error Space
+1	+1	$I \otimes I \otimes I$	$\boxed{\text{span}(\vec{0}\otimes\vec{0}\otimes\vec{0},\ \vec{1}\otimes\vec{1}\otimes\vec{1})}$ ← codespace
+1	−1	$I \otimes I \otimes X$	span$(\vec{0}\otimes\vec{0}\otimes\vec{1},\ \vec{1}\otimes\vec{1}\otimes\vec{0})$
−1	+1	$X \otimes I \otimes I$	span$(\vec{1}\otimes\vec{0}\otimes\vec{0},\ \vec{0}\otimes\vec{1}\otimes\vec{1})$
−1	−1	$I \otimes X \otimes I$	span$(\vec{0}\otimes\vec{1}\otimes\vec{0},\ \vec{1}\otimes\vec{0}\otimes\vec{1})$

Shorthand: $S_1 = ZZI$

$S_2 = IZZ$

S_1, S_2 are "check operators" for the code

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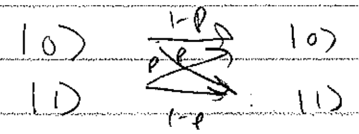
Quantum Error Correction

Binary symmetric

Not a unitary, will get to that soon

Bit-flip channel: w/prob p, X is applied to a qubit



$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ α, β unknown

$|\psi\rangle \to |\psi\rangle|\psi\rangle|\psi\rangle$? No-cloning! (Proof relied on linearity)

Solution: Use same circuit as for pbit repetition code!

Circuit trace:

do crossing out after next section

$\alpha|0\rangle + \beta|1\rangle \xrightarrow{\text{encode}} \alpha|000\rangle + \beta|111\rangle$

$\xrightarrow{\text{err}}$ $\alpha|000\rangle + \beta|111\rangle$ ~~prob~~ $\sqrt{(1-p)^3}$

$\alpha|001\rangle + \beta|110\rangle$
$\alpha|010\rangle + \beta|101\rangle$
$\alpha|100\rangle + \beta|011\rangle$ ~~prob~~ $\sqrt{(1-p)^2 p}(-i)$

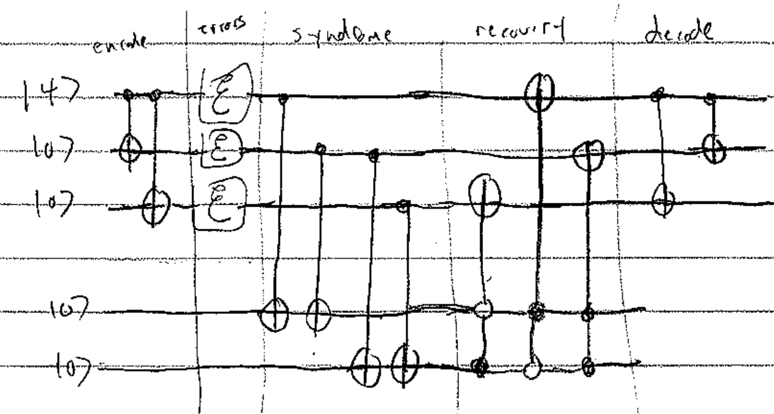
$\xrightarrow{\text{synd + recov + decode}}$ $\alpha|0\rangle + \beta|1\rangle$ prob $(1-p)^3 + 3p(1-p)^2$

$\alpha|110\rangle + \beta|001\rangle$
$\alpha|101\rangle + \beta|010\rangle$
$\alpha|011\rangle + \beta|100\rangle$ ~~prob~~ $\sqrt{(1-p)p^2}(-i)^2$

$\longrightarrow$ $\alpha|1\rangle + \beta|0\rangle$ prob $p^3 + 3(1-p)p^2$

$\alpha|111\rangle + \beta|000\rangle$ ~~prob~~ $\sqrt{p^3}(-i)^3$

Draw circuit first



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Challenges of QEC

1. Unitary errors (aka "coherent errors")

E.g., $|\psi\rangle \to U(\hat{x},\varepsilon)|\psi\rangle$ ("coherent bit flip channel")

$$= \left(I\cos\tfrac{\varepsilon}{2} - iX\sin\tfrac{\varepsilon}{2}\right)|\psi\rangle \qquad \text{Let } p := \sin^2\tfrac{\varepsilon}{2}$$

$$= \sqrt{1-p}\,|\psi\rangle - i\sqrt{p}\,X|\psi\rangle$$

Solution: Has same effect as BSC after syndrome measurement!

(Go back to table for BSC + put $\sqrt{\ }$'s around coeffs)

2. New kinds of unital channels ("prob p_i apply U_i")

Example:

Phase-flip channel: w/prob p, Z is applied to a qubit

prob $1-p$	prob p
$\|0\rangle \to \|0\rangle$	$\|0\rangle \to \|0\rangle$
$\|1\rangle \to \|1\rangle$	$\|1\rangle \to -\|1\rangle$

Repetition code makes it worse!

$\alpha|000\rangle + \beta|111\rangle \to \alpha|000\rangle - \beta|111\rangle$ if 1,3 phase flips occur

($\Rightarrow$ w/prob $3p(1-p)^2 + p^3 > p$!) (If $p<1/2$)

ZZI, IZZ measurements cannot detect phase!

Solution: Use $|\pm\rangle$ states!

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$|0\rangle \to |{+}{+}{+}\rangle = H^{\otimes 3}|000\rangle$
$|1\rangle \to |{-}{-}{-}\rangle = H^{\otimes 3}|111\rangle$

Why? Because $\left.\begin{array}{l} Z|+\rangle = |-\rangle \\ Z|-\rangle = |+\rangle \end{array}\right\}$ looks like a bit flip in +/− states!

Syndrome: parity of +/− states:

Aside: $\Pi_0 = |{+}{+}\rangle\langle{+}{+}| + |{-}{-}\rangle\langle{-}{-}|$,
$\Pi_1 = |{-}{+}\rangle\langle{-}{+}| + |{+}{-}\rangle\langle{+}{-}|$

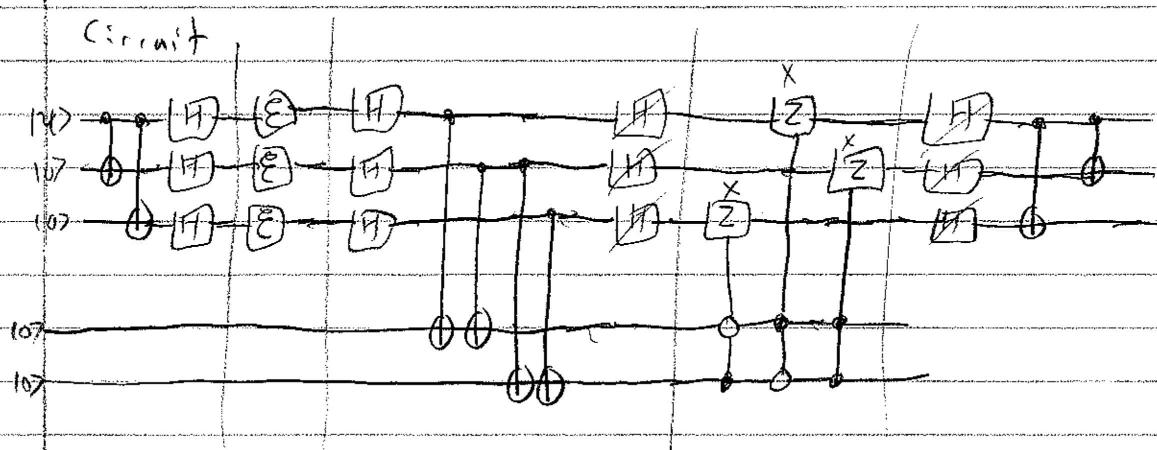
$(+1)\,\Pi_0 + (-1)\,\Pi_1 = X \otimes X$ (Exercise)

$\Rightarrow$ Measure $S_1 = XX1$, $S_2 = 1XX$

Recall $HX = ZH \Leftrightarrow HXH = Z,\ HZH = X$ (Matrix mult, not Tensor prod)

$\to$ Measure $\{H^{\otimes 3}(ZZ1)H^{\otimes 3},\ H^{\otimes 3}(1ZZ)H^{\otimes 3}\}$

Circuit



Note: Fails to correct bit flips!

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How to correct both bit-flips and phase-flips?

Solution: Concatenation!

Got this far

$$|0\rangle \to |{+}{+}{+}\rangle = \left[\tfrac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]^{\otimes 3} \to \left[\tfrac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\right]^{\otimes 3} \quad \text{(9 qubits)}$$

$$|1\rangle \to |{-}{-}{-}\rangle = \left[\tfrac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]^{\otimes 3} \to \left[\tfrac{1}{\sqrt{2}}(|000\rangle - |111\rangle)\right]^{\otimes 3}$$

Called the Shor code.

Syndrome observables: ZZI, IZZ within each block
$\bar{X}\bar{X}\bar{I}$, $\bar{I}\bar{X}\bar{X}$ between blocks

Aside: logical operators

Example 1: BFC $|0\rangle \to |000\rangle$, $|1\rangle \to |111\rangle$

What sends $\alpha|000\rangle + \beta|111\rangle$ to itself? ("Logical identity" $\bar{I}$)

$$e^{i\alpha}\{U(\hat{z},\theta)\otimes U(\hat{z},-\theta)\otimes I,\ I\otimes U(\hat{z},\theta)\otimes U(\hat{z},-\theta),\ U(\hat{z},\theta)\otimes I\otimes U(\hat{z},-\theta)\}$$

What sends $\alpha|000\rangle + \beta|111\rangle$ to $\alpha|111\rangle + \beta|000\rangle$? ("Logical $\bar{X}$")

$$\{\bar{I}(XXX),\ (XXX)\bar{I}\}$$

Many choices! Each is equivalent.

Our choice here: $\bar{I} = III$, $\bar{X} = XXX$.

("This freedom will be important later")