$\boxed{\text{Ph 452}}$ Lecture 13

[Q] Quick Review

Quantum Compiling: Solovay-Kitaev algorithm

Final remarks on universality:

Q: How to test if a gate set $\mathcal{G}$ is universal?

A: No general test - usually show any 2-qubit gate is approximatable.

Example: "Magic states" / Measurement-based QC

Gottesman-Knill Theorem: Quantum circuits over

$\mathcal{G} := \{H, S, CNOT\}$ can be efficiently simulated by classical circuits

Knill-Laflamme-Zurek basis:

$\mathcal{G} = \{H, S, CNOT, |T\rangle\}$ is universal,

$|T\rangle := \cos\frac{\pi}{8}|0\rangle + \sin\frac{\pi}{8}|1\rangle$

$S = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix}$

Relies on out-of-sequence lecture 12.

Proof sketch: Measure ZZ on $|\psi\rangle \otimes HS|T\rangle$, then apply CNOT:

$$|\psi\rangle\left[HS|T\rangle\right] = |\psi\rangle\left[\frac{1}{\sqrt{2}}\cos\frac{\pi}{8}(|0\rangle + |1\rangle) + \frac{i}{\sqrt{2}}\sin\frac{\pi}{8}(|0\rangle - |1\rangle)\right]$$

$$= |\psi\rangle\left[\frac{1}{\sqrt{2}}\left(e^{i\pi/8}|0\rangle + e^{-i\pi/8}|1\rangle\right)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\alpha e^{i\pi/8}|00\rangle + \alpha e^{-i\pi/8}|01\rangle + \beta e^{i\pi/8}|10\rangle + \beta e^{-i\pi/8}|11\rangle\right]$$

$$\rightarrow \begin{cases} \alpha e^{i\pi/8}|00\rangle + \beta e^{-i\pi/8}|11\rangle & \text{prob } \tfrac{1}{2} \\ \alpha e^{-i\pi/8}|01\rangle + \beta e^{+i\pi/8}|10\rangle & \text{"} \quad \text{"} \end{cases}$$

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$$\xrightarrow{CNOT} \begin{cases} \left[\alpha e^{i\pi/8}|0\rangle + \beta e^{-i\pi/8}|1\rangle\right] \otimes |0\rangle & p = 1/2 \\ \left[\alpha e^{-i\pi/8}|0\rangle + \beta e^{+i\pi/8}|1\rangle\right] \otimes |1\rangle & p = 1/2 \end{cases}$$

$$= \begin{cases} T|\psi\rangle \otimes |0\rangle & p = 1/2 \\ T^{\dagger}|\psi\rangle \otimes |1\rangle & p = 1/2 \end{cases}$$

Factoid: Random walk in 1D visits every integer with probability 1.

$\Rightarrow$ Can apply T gate $\square$

Factoid: 1-qubit measurements + a "cluster state" on $n \times T$ qubits suffice to simulate any n-qubit T-gate circuit!

[Raussendorf, Briegel, Phys. Rev. Lett. 86, 5188 (2001)]

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Grover's Algorithm

N-bit OR: $N \in \mathbb{N}$, $\Sigma = \mathbb{B}$, $Y = \mathbb{B}$, $X = \Sigma^N$ $g(x) = \bigwedge_{i=0}^{N-1} x_i$

Just as hard for

$$X = \{ x \in \Sigma^N \mid wt(x) = 0 \text{ or } wt(x) = 1 \}$$

$$= \{ x \in \Sigma^N \mid x_i = \begin{cases} 1 & i = w \\ 0 & i \neq w \end{cases} \}$$

No harder than computing

$$g(x) = w \qquad (w \text{ for "winner"}) \qquad (Y = \mathbb{B}^n,\ n = \lceil \log_2 N \rceil)$$

Definitions:

Oracle: $U_x : \underbrace{|i\rangle}_{n \text{ bits}} \underbrace{|j\rangle}_{1 \text{ bit}} \mapsto |i\rangle |j \oplus x_i\rangle$

(Recall $U_x |i\rangle |-\rangle \mapsto (-1)^{x_i} |i\rangle |-\rangle$)

Define $Z_x |i\rangle \mapsto (-1)^{x_i} |i\rangle$

$$Z_x = I - 2|w\rangle\langle w|$$

Define $Z_0 := I - 2 \, |0\rangle \underbrace{\langle 0|}_{n\text{-bits}}$

Recall $S := H^{\otimes n} Z_0 H^{\otimes n} = I - 2|s\rangle\langle s|$, $\quad |s\rangle := \frac{1}{\sqrt{N}} \sum_{i=0}^{N-1} |i\rangle$

Define $G := -S Z_x$

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Grovers algorithm:

① Apply $G^k H^{\otimes n}|0\rangle^{\otimes n}$, $k = \lfloor \frac{\pi}{4}\sqrt{N} \rfloor$

② Measure the n qubits, obtaining $|t\rangle$

③ Check that $Z_x|t\rangle = -|t\rangle$. If not, repeat.

Analysis 1: Algebraic

$$H^{\otimes n}|0\rangle^{\otimes n} = |s\rangle = \frac{1}{\sqrt{N}}\sum_{i=0}^{N-1}|i\rangle$$

$$= \sqrt{\frac{N-1}{N}}\underbrace{\left(\frac{1}{\sqrt{N-1}}\sum_{i\neq w}|i\rangle\right)}_{|s'\rangle} + \frac{1}{\sqrt{N}}|w\rangle$$

$$Z_x|s'\rangle = |s'\rangle$$

$$Z_x|w\rangle = -|w\rangle$$

$$-S|s'\rangle = (2|s\rangle\langle s| - I)|s'\rangle$$

$$= 2\sqrt{\frac{N-1}{N}}|s\rangle - |s'\rangle$$

$$= \left(2\left(\frac{N-1}{N}\right) - 1\right)|s'\rangle + \frac{2\sqrt{N-1}}{N}|w\rangle$$

$$= -\left(\frac{2}{N} - 1\right)|s'\rangle + \frac{2\sqrt{N-1}}{N}|w\rangle$$

$$-S|w\rangle = (2|s\rangle\langle s| - I)|w\rangle$$

$$= 2\left(\frac{1}{\sqrt{N}}\right)|s\rangle - |w\rangle$$

$$= \frac{2\sqrt{N-1}}{N}|s'\rangle + \left(\frac{2}{N} - 1\right)|w\rangle$$

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$-S|w\rangle = -\cos\theta|w\rangle + \sin\theta|s'\rangle$

$-S|s'\rangle = \sin\theta|w\rangle + \cos\theta|s'\rangle$

$\sin\theta = \frac{2\sqrt{N-1}}{N}, \quad \cos\theta = \frac{2}{N} - 1$

$G|w\rangle = \cos\theta|w\rangle - \sin\theta|s'\rangle$

$G|s'\rangle = \sin\theta|w\rangle + \cos\theta|s'\rangle$

$G^k|w\rangle = \cos k\theta|w\rangle - \sin k\theta|s'\rangle$

$G^k|s'\rangle = \sin k\theta|w\rangle + \cos k\theta|s'\rangle$

Start with

$$|\psi\rangle = \underbrace{\sqrt{\frac{N-1}{N}}|s'\rangle}_{\text{most support here}} + \frac{1}{\sqrt{N}}|w\rangle$$

If $\sin k\theta = 1$, will swap the amplitudes!

Could numerically solve, but easier to note

$\sin\theta \approx \theta$ for small θ (Taylor expansion)

$\Rightarrow \theta \approx \frac{2\sqrt{N-1}}{N}$ for large N

$\Rightarrow k = \frac{\pi}{2}\frac{N}{2\sqrt{N-1}} \approx \frac{\pi}{4}\sqrt{N}$!

→ Analysis of classical post-processing omitted

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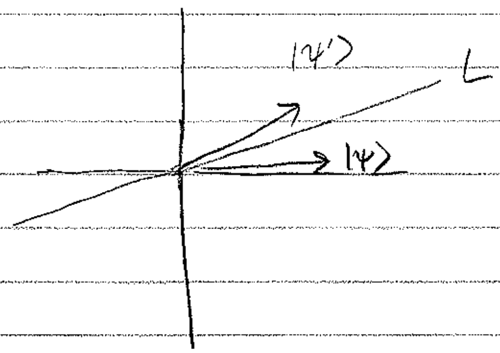
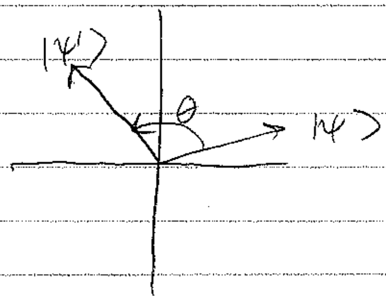
Analysis 2: Geometric

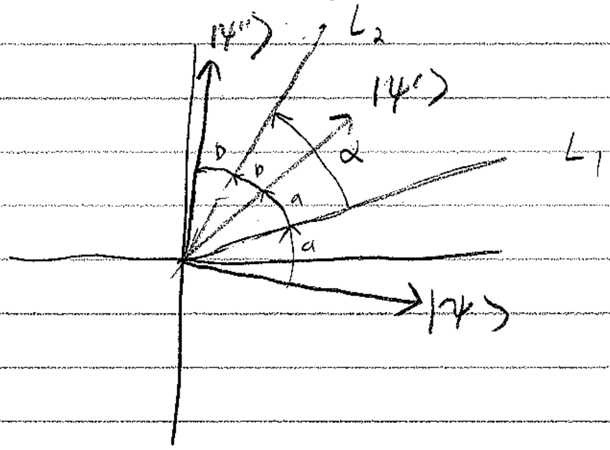


Reflection about L



Rotation by θ

Factoid: Let L_1, L_2 be lines separated by angle α. Then reflecting about L_1, then L_2 is the same as rotating by 2α:



Factoid: $Z_x = I - 2|w\rangle\langle w|$, $S = I - 2|s\rangle\langle s|$ are reflections about $|w^\perp\rangle$, $|s^\perp\rangle$

Got this far