- HW - Please turn in
- Next HW due in a week - only 4 probs
- No class Thu, Tue. Maybe Thu. after

Ph 452 Lecture 17

Quantum Error Correction: "I am QEC, so can you" - Paraphrasing Stephen Colbert.

IQI Quick Review: Lectures 10, 12

- Repetition code: $0 \to 000$, $1 \to 111$ $\quad p_{err} \to O(p_{err}^2)$ in binary symmetric channel
- No-cloning theorem: pbits, qubits
- Pbit repetition code: Measure syndrome, apply recovery
- Observables in QM: $A = \underset{\text{"value"}}{\sum f(i)} \underset{\text{projector}}{\Pi_i}$ — Hermitian
- Qubit repetition code: syndrome observables ZZI, IZZ, recoveries XII, IXI, IIX
- New quantum errors
  - Small unitary errors, e.g. $U = \sqrt{1-p}\,I - i\sqrt{p}\,X$, p small
    - $\to$ Syndrome measurement makes them big, but with small probability
  - Phase-flip channel: Apply Z with prob. p
    - $\to$ Measure syndrome observables XXI, IXX instead
    - $|0\rangle \to |{+}{+}{+}\rangle$, $|1\rangle \to |{-}{-}{-}\rangle$

Conundrum:

Bit-flip channel: X w/prob. p_x

Phase-flip channel: Z w/prob p_z

$|0\rangle \to |000\rangle$ $\quad$ maps $p_x \to O(p_x^2)$ $\quad$ "Bit-flip code"

$|1\rangle \to |111\rangle$ $\quad$ $p_z \to 3p_z + O(p_z^2)$

$|0\rangle \to |{+}{+}{+}\rangle$ $\quad$ maps $p_x \to 3p_x + O(p_x^2)$ $\quad$ "Phase-flip code"

$|1\rangle \to |{-}{-}{-}\rangle$ $\quad$ $p_z \to O(p_z^2)$

Can a quantum code correct both bit- and phase-flips?

Yes! Shor code

Ph 452 Lecture 17

$|0\rangle \to |\bar{0}\rangle = \frac{1}{2^{3/2}}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)(|000\rangle + |111\rangle)$

$|1\rangle \to |\bar{1}\rangle = \frac{1}{2^{3/2}}(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)(|000\rangle - |111\rangle)$

To detect a bit flip:

- Measure parities of neighboring qubits in a block

To detect a phase flip:

- Measure parities of <u>phases</u> of neighboring blocks

Curious feature: Z_1 can be corrected by Z_2 or Z_3

$\Rightarrow$ Shor code is a <u>"degenerate code"</u>

Shor code is a <u>"concatenated code"</u>

First phase-flip code:

$|0\rangle \to |+++\rangle$

$|1\rangle \to |---\rangle$

Then bit-flip code:

$|+++\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]^{\otimes 3} \to \left[\frac{1}{\sqrt{2}}|000\rangle + |111\rangle\right]^{\otimes 3}$

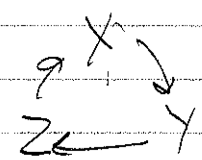
$|---\rangle = \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]^{\otimes 3} \to \left[\frac{1}{\sqrt{2}}|000\rangle - |111\rangle\right]^{\otimes 3}$

$\boxed{\text{Ph 452}}$ Lecture 17

Shor code can correct a simultaneous X and Z error, <u>even on the same qubit</u>!

$$\text{E.g.}\quad X_1 Z_1 |\bar{1}\rangle = \frac{1}{2^{3/2}}\big(\underbrace{|1}_{\text{qubit parity amiss}}00\rangle + |011\rangle\big)\overbrace{\big(|000\rangle - |111\rangle\big)\big(|000\rangle - |111\rangle\big)}^{\text{block phase parity amiss}}$$

Mathematical factoid:



<u>Clockwise:</u>	<u>CCW:</u>
$XY = iZ$	$YX = -iZ$
$YZ = iX$	$XZ = -iY$
$ZX = iY$	$ZY = -iX$

Example: $XZ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -iY$

$\Rightarrow$ Shor code can correct $\{I, X, Y, Z\}$ on any qubit

(Overall phase of $\pm i$ is unphysical in $XZ = -iY$, e.g.)

Most general 1-qubit error:

$|\psi\rangle \to E|\psi\rangle$,

Even more general than allowed by physics. e.g. Not a unitary or a projector

$E = \alpha_0 I + \alpha_1 X + \alpha_2 Y + \alpha_3 Z \quad , \alpha_i \in \mathbb{C}$

"Pauli Matrices are a <u>basis</u> for 2x2 matrices over $\mathbb{C}$"

$\boxed{Ph 452}$ Lecture 17

$|\psi\rangle \to |\bar{\psi}\rangle$ (encoding)

$\to E|\bar{\psi}\rangle$ (error)

$\to \alpha_0 I|\bar{\psi}\rangle|I \text{ syndrome}\rangle$
$+\alpha_1 X|\bar{\psi}\rangle|X \text{ syndrome}\rangle$ ("coherent" syndrome measurement)
$+\alpha_2 Y|\bar{\psi}\rangle|Y \text{ syndrome}\rangle$
$+\alpha_3 Z|\bar{\psi}\rangle|Z \text{ syndrome}\rangle$ ("could measure & get σ_i acting w/ prob $|\alpha_i|^2$")

$\to |\bar{\psi}\rangle(\alpha_0|I \text{ syn}\rangle$
$+\alpha_1|X \text{ syn}\rangle$ ("coherent" recovery)
$+\alpha_2|Y \text{ syn}\rangle$
$+\alpha_3|Z \text{ syn}\rangle)$

"Degeneracy again: $Z_1 Z_3 = I$ on codewords"

$\Rightarrow$ Any 1-qubit error can be corrected!

Idea: Information about errors reversibly moved to ancilla qubits.

Note: There are many errors Shor code can't correct:

E.g. $X_1 X_2$ or $Z_3 Y_5 X_6$, etc. $= Z_3(-iZ_5X_5)X_6 = -i\underbrace{Z_3 Z_5}_{\text{problem}}\underbrace{X_5 X_6}_{\text{problem}}$

In a realistic error model, each qubit in error w/ prob p

QEC maps $p \to O(p^2)$, just like Classical error correction.

Lingering Questions:

(1) What is the most general error channel allowed by QM?

not just "apply U_i w/ prob p_i."

(2) What other quantum error correcting codes exist?

"Spaghetti monster Theory: part of a spaghettum $\neq$ a spaghettum"

"QM: The whole is greater than the sum of its parts! (Entanglement/Local realism)"

"That all states can be thought of as part of a whole is a physical fact, not a mathematical fact. Some day a spaghetti-o might be discovered."

Ph 432 | Lecture 17

Open Quantum Systems

QM framework:

Correctly describes part of a larger system?

① States: $|\psi\rangle \cong e^{i\theta}|\psi\rangle \in \mathbb{C}_d^{\otimes n}$ — FALSE!

② Dynamics: $U : U^\dagger U = I$ — FALSE!

③ Measurements: $\{\Pi_i\} : \Pi_i^\dagger = \Pi_i,\ \Pi_i \Pi_j = \delta_{ij}\Pi_j$ — FALSE!

States:

$$|\psi\rangle_{AB} = \sum_{i,j} \alpha_{ij} |i\rangle_A |j\rangle_B \quad , \quad \alpha_{ij} \in \mathbb{C}, \quad \sum_{i,j} |\alpha_{ij}|^2 = 1$$

Want to have object describing Alice's description of the state.

Fact 1: Schmidt decomposition

Any bipartite state can be written as

$$|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |i\rangle_B$$

where $p_i \in [0,1]$ and $\{|i\rangle_A\}, \{|i\rangle_B\}$ are orthonormal sets of states on A & B (not nec. computational basis states)

Ph 432 Lecture 17

Proof: Let $U := \sum U_{k\ell} |k\rangle\langle \ell|$ on A

$V := \sum V_{mn} |m\rangle\langle n|$ on B

Then

$$U \otimes V \sum_{ij} \alpha_{ij} |i\rangle|j\rangle = \sum_{\substack{ijk\ell \\ mn}} U_{k\ell}\, \alpha_{ij}\, V_{mn}\, |k\rangle \underbrace{\langle \ell | i\rangle}_{\delta_{\ell i}} |m\rangle \underbrace{\langle n | j\rangle}_{\delta_{nj}}$$

$$= \sum_{k\ell mn} U_{k\ell}\, \alpha_{\ell n}\, V_{mn}\, |k\rangle|m\rangle$$

$$= \sum_{km} \alpha'_{km} |k\rangle|m\rangle$$

where

$$\alpha'_{km} = \sum_{\ell n} U_{k\ell}\, \alpha_{\ell n}\, V_{mn} \iff \alpha' = U \alpha V^T$$

By singular value decomposition, choose U, V^T that diagonalize α into a positive real diagonal matrix α' $\square$

$\left(|\psi\rangle_{AB} = \sum \sqrt{p_i}\, |i\rangle_A |i\rangle_B \quad b/c \quad \sum |\sqrt{p_i}|^2 = 1. \right)$

Ambiguity in Schmidt decomposition

If $|\psi\rangle_{AB} = \sum_i \sqrt{p_i}\, |i\rangle_A |i\rangle_B$, then

$$|\psi\rangle_{AB} = \sum_{ijk} \sqrt{p_i}\, U^*_{ij}\, |j\rangle_A\, U_{ik} |k\rangle_B$$

for any unitary matrix U

Proof Next time