Lecture 17
Quantum Error Correction: "I am GEC, so can you" - P. W. Shor
1Q1 Quick Review: Lecture 10, 12

- Repetition code: 0 000, 1 111, 0 00 0 00 or 0 00 (free) in binary symmetric channel
- No-cloning theorem: phits, qubits
- Phit repetition code: Measure syndrome, apply recovery
- Observables in QM: \( A = \sum E_i \Pi_i \) = Hermatian "value" projector
- Qubit repetition code: Syndrome observables \( Z \), \( Z Z \), \( X X \), \( X \), \( Z \), \( X X \), \( X \), \( Z \)
- New quantum errors
  - Small unitary errors, e.g., \( U = \exp(i \pi \hat{X}) \), \( \hat{X} \) small
    \( \rightarrow \) syndrome measurement makes them big, but with small probability
  - Phase-flip channel: Apply \( Z \) with prob \( p \)
    \( \rightarrow \) Measure syndrome observables \( X X \), \( X \), \( X \) instead
    \( 10 \rightarrow + + + , \ 11 \rightarrow - - - \)

Conundrum:

- Bit-flip channel: \( X \) w/ prob \( p_x \)
- Phase-flip channel: \( Z \) w/ prob \( p_z \)

\[ 10 \rightarrow 0000 \] maps \( p_x \rightarrow O(p_x^2) \) "Bit-flip code"
\[ 11 \rightarrow 1111 \] \( p_z \rightarrow 3 p_z + O(p_z^2) \)

\[ 10 \rightarrow + + + \] maps \( p_x \rightarrow 3 p_x + O(p_x^2) \) "Phase-flip code"
\[ 11 \rightarrow - - - \] \( p_z \rightarrow O(p_z^2) \)

Can a quantum code correct both bit and phase-flips?

Yes! Shor code
\[ \begin{align*}
\psi &= \frac{1}{\sqrt{2}} (1000 + 1111) \times (1000 + 1111) \\
10\rangle &\rightarrow 11\rangle = \frac{1}{2^{3/2}} (1000 - 1111) \times (1000 - 1111)
\end{align*} \]

To detect a bit flip:

- Measure parities of neighboring qubits in a block.

To detect a phase flip:

- Measure parities of phases of neighboring blocks.

Curious feature: \( Z_2 \) can be corrected by \( Z_3 \) or \( Z_3 \).

\( Z \) & \( Z \) code is a "degenerate code."

\( Z \) & \( Z \) code is a "concatenated code."

First phase-flip code:

\[ \begin{align*}
10\rangle &\rightarrow 1+++ \\
11\rangle &\rightarrow 1--
\end{align*} \]

Then bit-flip code:

\[ \begin{align*}
+++ &\rightarrow \left[ \frac{1}{\sqrt{2}} (1000 + 1111) \right]^{\otimes 3} \rightarrow \left[ \frac{1}{\sqrt{2}} (1000 + 1111) \right]^{\otimes 3} \\
--- &\rightarrow \left[ \frac{1}{\sqrt{2}} (1000 - 1111) \right]^{\otimes 3} \rightarrow \left[ \frac{1}{\sqrt{2}} (1000 - 1111) \right]^{\otimes 3}
\end{align*} \]
Shor code can correct a simultaneous X and Z error, even on the same qubit!

$$|XZ, 11\rangle = \frac{1}{\sqrt{2}} (|100\rangle + |111\rangle)(|000\rangle - |111\rangle)$$

Example:

- Clockwise:
  - $XY = iZ$
  - $YZ = iX$
  - $ZX = -iY$

- Counterclockwise:
  - $YX = -iZ$
  - $XZ = -iY$
  - $ZY = -iX$

Example:

$$XZ = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ 0 & 1 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = -iY$$

Shor code can correct $X$, $Y$, $Z$ on any qubit.

Overall phase of $Z$ is unphysical in $XZ = -iY$, e.g.

Most general 1-qubit error:

$$|11\rangle \rightarrow E |11\rangle$$

$$E = x_0 I + x_1 X + x_2 Y + x_3 Z$$

"Pauli Matrices are a basis for $2 \times 2$ matrices over C."
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\[ |\psi\rangle \rightarrow |\overline{\phi}\rangle \quad \text{(encoding)} \]
\[ E(\overline{\phi}) \quad \text{(error)} \]
\[ \rightarrow 8y (|\overline{\phi}\rangle |I\rangle \text{ syndrome}) \]
\[ + a_1 |x\rangle |x\text{ syndrome} \] \quad \text{("coherent" syndrome measurement)}
\[ + a_2 |y\rangle |y\text{ syndrome} \]
\[ + a_3 |z\rangle |z\text{ syndrome} \] \quad \text{("could measure + get or acting up prob } |a_1|^2 \text{")}

\[ \rightarrow |\overline{\phi}\rangle (x_0 |I\rangle \text{ syn}) \]
\[ + a_1 |x\rangle |x\text{ syn} \] \quad \text{("coherent" recovery)}
\[ + a_2 |y\rangle |y\text{ syn} \]
\[ + a_3 |z\rangle |z\text{ syn} \]

\[ \Rightarrow \text{Any 1-qubit error can be corrected!} \]

Idea: Information about errors reversibly moved to ancilla qubits.

Note: There are many errors. Short code can't correct:

E.g., \( X_1 X_2 \) or \( Z_3 Y_5 X_6 = Z_3 (Z_5 X_5) X_6 = Z_3 Z_5 X_5 X_6 \)

In a realistic error model, each qubit in error with prob \( p \)

QECC maps \( p \rightarrow O(p^2) \) just like classical error correction.

Listening questions:
1. What is the most general error channel allowed by QECC?
2. What other quantum error correcting codes exist?
Open Quantum Systems

QM framework:

1. States: \( \mathcal{H} = \mathbb{C}^n \)

2. Dynamics: \( U : U^* U = I \)

3. Measurements: \( \tau_i \beta : \tau_i^* = \tau_i, \tau_i \beta \tau_i^* = \delta_i \beta_i \)

States:

\[ |\psi\rangle = \sum_{i,j} a_{ij} |i\rangle_A |j\rangle_B, \quad a_{ij} \in \mathbb{C}, \quad \sum_{i,j} |a_{ij}|^2 = 1 \]

Want to have object describing Alice's description of the state.

Fact 1: Schmidt decomposition

Any bipartite state can be written as

\[ |\psi\rangle_{AB} = \sum \sqrt{p_i} |i\rangle_A |j\rangle_B \]

where \( p_i \in [0,1] \) and \( |i\rangle_A, |j\rangle_B \) are orthonormal sets of states on \( A \) and \( B \) (not necessarily computational basis states).
Proof: Let \( \mathbf{U} = \sum U_{kl} |k\rangle |l\rangle \) on \( A \)
\[
\mathbf{V} = \sum V_{mn} |m\rangle |n\rangle \) on \( B \)

Then
\[
\mathbf{U} \otimes \mathbf{V} = \sum_{ij} \alpha_{ij} |i\rangle |j\rangle
= \sum_{km} U_{kl} \alpha_{km} V_{mn} |k\rangle |m\rangle
= \sum_{km} \sum_{ij} U_{kl} \alpha_{km} V_{mn} |j\rangle |k\rangle
\]

where
\[
\alpha'_{km} = \sum_{in} U_{kn} \alpha_{im} V_{mn} \quad \Rightarrow \quad \alpha' = U \alpha V^T
\]

By singular value decomposition, choose \( U, V^\dagger \) that diagonalize \( \alpha \) into a positive real diagonal matrix \( \alpha' \)

Ambiguity in Schmidt decomposition:
If \( |\psi\rangle_{ab} = \sum \sqrt{p_i} |i\rangle_a |j\rangle_b \), then
\[
|\psi\rangle_{ab} = \sum \sqrt{p_i} U_{ij} |i\rangle_a |k\rangle_b
\]
For any unitary matrix \( U \)

Proof Next time