Ph452 Lecture 18

QEC: Know thy enemy

Open Quantum systems: new objects for states, dynamics, measurements

States:

Schmidt decomposition:

$$|\psi\rangle = \sum \alpha_{ij} |i\rangle_A |j\rangle_B$$

$$= \sum_{ijk} U_{ik} D_{kk} V_{kj} |i\rangle|j\rangle \qquad (\text{By SVD})$$

$$= \sum_k \underbrace{D_{kk}}_{\sqrt{p_k}} \underbrace{\left(\sum_i U_{ik}|i\rangle\right)}_{|\psi_k\rangle_A} \underbrace{\left(\sum_j V_{kj}|j\rangle\right)}_{|\varphi_k\rangle_B}$$

Orthonormal Schmidt bases

$p_k \geq 0$ (by SVD, $D \geq 0$)

$\sum p_k = 1$ (by $\langle\psi|\psi\rangle = 1$)

$\langle\psi_j|\psi_k\rangle = \langle\varphi_j|\varphi_k\rangle = \delta_{jk}$ (by $U^\dagger U = V^\dagger V = I$ and ${}_A\langle j|k\rangle_A = {}_B\langle j|k\rangle_B = \delta_{jk}$)

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<u>States:</u> Open quantum systems

A = part of Q. system
B = rest of the universe

$$|\Psi\rangle_{AB} = \sum \sqrt{p_k}\, |k\rangle_A |k\rangle_B$$ by Schmidt decomp

Measurement on A: $\{\Pi_i^A \otimes I^B\}$, $\Pi_i \Pi_j = \delta_{ij} \Pi_j$, $\sum \Pi_i = I$

$$\begin{aligned}
P_r(k) &= {}_{AB}\langle\Psi|\, \Pi_k \otimes I \,|\Psi\rangle_{AB} \\
&= \sum_{ij} \sqrt{p_i p_j}\; {}_A\langle i|\Pi_k|j\rangle_A \; \underbrace{{}_B\langle i|j\rangle_B}_{\delta_{ij}} \\
&= \sum_i p_i \langle i|\Pi_k|i\rangle \\
&= \sum_{ij} p_i \langle i|j\rangle\langle j|\Pi_k|i\rangle \qquad \text{Insert identity!} \\
&= \sum_{ij} \langle j|\Pi_k\, p_i |i\rangle\langle i|j\rangle \\
&= \sum_{ij} \langle j|\, \Pi_k \left(p_i |i\rangle\langle i|\right) |j\rangle \\
&= \mathrm{tr}(\Pi_k \rho_A), \quad \rho_A := \sum_i p_i |i\rangle\langle i|
\end{aligned}$$

ρ_A is the density matrix describing the "reduced state" of system A.

$$\rho_A = \mathrm{tr}_B\left(|\Psi\rangle_{AB}\,{}_{AB}\langle\Psi|\right) = \sum {}_B\langle i|\Psi\rangle_{AB}\,{}_{AB}\langle\Psi|i\rangle_B$$

$$= \sum p_i\, |i\rangle_A\,{}_A\langle i|$$

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Features:

$$\mathrm{Tr}\,\rho = \sum_{ij} \langle j | p_i | i \rangle \overbrace{\langle i | j \rangle}^{\delta_{ij}} = \sum_i p_i = 1$$

$$\rho^\dagger = \sum p_i^* (|i\rangle\langle i|)^\dagger = \sum p_i |i\rangle\langle i| = \rho$$

$\rho \geq 0$: ρ has positive eigenvalues $p_i \geq 0$

Factoid: Any matrix satisfying $\mathrm{tr}\,\rho = 1$, $\rho \geq 0$, $\rho = \rho^\dagger$ is a valid density matrix

pure state: only one p_i : e.g. $\rho = |0\rangle\langle 0|$ or $\rho = |+\rangle\langle +|$.
mixed state: otherwise

Examples:
$$\rho_1 = |-\rangle\langle -| = \left(\tfrac{1}{\sqrt{2}}\right)(|0\rangle - |1\rangle)\left(\tfrac{1}{\sqrt{2}}\right)(\langle 0| - \langle 1|)$$

$$= \tfrac{1}{2}\left(|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|\right)$$

$$= \tfrac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\rho_2 = |+\rangle\langle +| = \tfrac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\rho_3 = \tfrac{1}{2}|0\rangle\langle 0| + \tfrac{1}{2}|1\rangle\langle 1| = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho_4 = \tfrac{1}{4}|0\rangle\langle 0| + \tfrac{3}{4}|+\rangle\langle +| = \begin{pmatrix} 5/8 & 3/8 \\ 3/8 & 3/8 \end{pmatrix}$$

more later

$$\rho_5 = \tfrac{1}{2}|+\rangle\langle +| + \tfrac{1}{2}|-\rangle\langle -| = \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \rho_3 \,!$$

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Ensemble interpretation:

$\rho = \sum p_i |i\rangle\langle i| \iff \{p_i, |i\rangle\}$

Why? For an ensemble, measuring $\{\Pi_k\}$ yields

$Pr(k||i\rangle) = \langle i|\Pi_k|i\rangle, \quad Pr(k) = \sum p_i \langle i|\Pi_k|i\rangle = tr(\Pi_k \rho)$

How? Measure B system in Schmidt basis:

$I^B \otimes \Pi_i^B, \quad \Pi_i^B = |i\rangle\langle i|$

$$\frac{(I \otimes \Pi_i) \sum_k \sqrt{p_k}\, |k\rangle_A |k\rangle_B}{\sqrt{Pr(i)}} = \frac{\sqrt{p_i}\, |i\rangle_A}{\sqrt{p_i}} |i\rangle$$

$\rightarrow \{p_i, |i\rangle_A\}$

Purifications:

Let $\rho = \sum p_i |\psi_i\rangle\langle\psi_i|$ (some $p_i = 0$, possibly)

Let $\{|\varphi_i\rangle_B\}$ be a basis for B, then

$|\psi\rangle_{AB} = \sum \sqrt{p_i}\, |\psi_i\rangle|\varphi_i\rangle$ is a purification of ρ_A.

Purification ambiguity! Get same ρ_A from

$$\left.\begin{aligned} |\Phi_1\rangle_{AB} &= \sum \sqrt{p_i}\, |i\rangle|j\rangle \\ |\Phi_2\rangle_{AB} &= \sum \sqrt{p_i}\, |i\rangle|j'\rangle \end{aligned}\right\} \quad |\Phi_1\rangle_{AB} = (I \otimes U_B)\, |\Phi_2\rangle_{AB}$$

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Ensemble ambiguity $\rho = \{p_i, |\psi_i\rangle\}$, $\sigma = \{q_i, |\tilde{\psi}_i\rangle\}$

$$\rho \xrightarrow{\text{purify}} \sum \sqrt{p_i}\, |\psi_i\rangle|\varphi_i\rangle = |\Psi\rangle_{AB}$$

$$\sigma \xrightarrow{\text{purify}} \sum \sqrt{q_i}\, |\tilde{\psi}_i\rangle\, |\tilde{\varphi}_i\rangle = |\tilde{\Psi}\rangle_{AB}$$

If $\rho = \sigma$, can pad purifications so bases same dimension.

$$\Rightarrow \quad |\tilde{\varphi}_i\rangle = U_B|\varphi_i\rangle \quad \text{for some } U_B^\dagger U_B = I_B$$

$$|\tilde{\Psi}\rangle_{AB} = \sum_i \sqrt{q_i}\, |\tilde{\psi}_i\rangle \otimes U|\varphi_i\rangle$$ matrix

numbers

$$= \sum \sqrt{q_i}\, |\tilde{\psi}_i\rangle \otimes \Big(\sum_j U_{ji} |\varphi_j\rangle\Big)$$

$$= \sum_{ij} U_{ji} \sqrt{q_i}\, |\tilde{\psi}_i\rangle |\varphi_j\rangle$$

$\Rightarrow$ $\{p_i, |\psi_i\rangle\}$ and $\{q_i, |\tilde{\psi}_i\rangle\}$ represent same density matrix if

$$\boxed{\sqrt{p_i}\, |\psi_i\rangle = \sum_{j=1}^{d} \sqrt{q_j}\, U_{ij}\, |\tilde{\psi}_j\rangle}$$

Example:

$H|0\rangle = |+\rangle$, $\quad H|1\rangle = |-\rangle$

$\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}|+\rangle\langle +| + \frac{1}{2}|-\rangle\langle -|$ can be seen from

$$\sqrt{\tfrac{1}{2}}\, |+\rangle = \sqrt{\tfrac{1}{2}}\, H|0\rangle, \quad \sqrt{\tfrac{1}{2}}\, |-\rangle = \sqrt{\tfrac{1}{2}}\, H|1\rangle.$$

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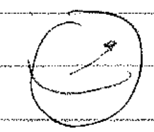
Bloch Ball

Recall Bloch sphere:

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi$$



Work out trig at home for yourself. →

Then $|\psi\rangle\langle\psi| = \frac{1}{2}\left[I + Z\cos\theta + X\sin\theta\cos\varphi + Y\sin\theta\sin\varphi\right]$

$$= \frac{1}{2}(I + \vec{p}\cdot\vec{\sigma}), \quad |\vec{p}| = 1$$

Generally, any 2×2 matrix can be written

$$\rho = c_I I + c_x X + c_y Y + c_z Z, \quad c_i \in \mathbb{C}$$

$\rho = \rho^\dagger \iff c_i = c_i^*$ (the c_i are real.)

$\mathrm{tr}\,\rho = 1 \iff c_I = 1/2$ ($\mathrm{tr}\,\sigma_i = 0$, $\mathrm{tr}\,I = 2$)

$\rho \ge 0 \iff \det\rho \ge 0$ (because $\mathrm{tr}\,\rho = 1$, at most 1 neg. eigenval)

$$\begin{vmatrix} \frac{1}{2} + c_z & c_x + ic_y \\ c_x - ic_y & \frac{1}{2} - c_z \end{vmatrix} = \frac{1}{4} - c_z^2 - c_x^2 - c_y^2 \ge 0$$

$$c_x^2 + c_y^2 + c_z^2 \le 1/4$$

$$\Rightarrow \boxed{\rho = \frac{1}{2}(I + \vec{p}\cdot\vec{\sigma}), \quad |\vec{p}| \le 1}$$

Bloch ball includes interior of Bloch sphere!

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Next time: Open systems dynamics & evolution.