10.1 Quick review

$|\psi\rangle \rightarrow \rho = \sum \lambda_i |\psi_i\rangle \langle \psi_i|$

$tr \rho = 1, \rho = \rho^+, \rho \geq 0$

Dynamics:

$|\psi\rangle \rightarrow U\langle \psi\rangle \Rightarrow \rho = \sum \lambda_i |\psi_i\rangle \langle \psi_i|$

$\langle \psi_i | \langle \psi_i| U \rangle \rightarrow \sum \lambda_i U |\psi_i \rangle \langle \psi_i| U^*$

$\Rightarrow U \rho U^*$

Measurement:

$|\psi\rangle \rightarrow \frac{\Pi_k |\psi\rangle}{\sqrt{\text{prob}(|k\rangle)}} \Rightarrow \rho = \sum \lambda_i |\psi_i\rangle \langle \psi_i|$  \(\langle \psi_i | \Pi_k \rangle \langle \psi_i | \Pi_k \rangle \)

But $\text{prob}(|k\rangle) = tr(\Pi_k \rho), \text{ so}$

$\rho \rightarrow \frac{\Pi_k \rho \Pi_k}{\text{tr}(\Pi_k \rho)}$ \(w/\text{prob} \mid tr(\Pi_k \rho) \text{ "Conditional evolution"}

Disregarding outcome label $\Rightarrow$ get an ensemble average!

$\rho \rightarrow \sum \text{prob}(|k\rangle) \frac{\Pi_k \rho \Pi_k}{\text{tr}(\Pi_k \rho)} = \sum \Pi_k \rho \Pi_k$

"Unconditional evolution"
Open System dynamics

Recall \( \rho_A = \text{tr}_B |\psi_{AB}\rangle \langle \psi_{AB}| \).

Partial trace averages over outcomes of auxiliary system.

\[
|\psi_{AB}\rangle = \frac{1}{\sqrt{\delta}} (|00\rangle + |11\rangle)
\]

\[
\text{tr}_B |\psi_{AB}\rangle \langle \psi_{AB}| = \frac{1}{\delta} \left( \langle 00 | \left( \frac{1}{\sqrt{\delta}} \right) (|00\rangle + |11\rangle) \langle 00 \rangle + \langle 11 | \left( \frac{1}{\sqrt{\delta}} \right) (|00\rangle + |11\rangle) |11\rangle \right)
\]

\[
= \frac{1}{\delta} \left( |00\rangle \langle 00 | + |11\rangle \langle 11 | \right)
\]

Quantum Operation:

\[
\rho_A \rightarrow \text{tr}_B \left( U_{AB} \rho_A \otimes \sigma_B U_{AB}^+ \right) \quad \text{"Unitary representation"}
\]

Wlog \( \sigma_B = |\psi_{B}\rangle \langle \psi_{B}| \) ("Church of the larger Hilbert space")

\[
\rho_A \rightarrow \sum_B \langle \psi_B | U_{AB} \rho_A \otimes |\psi_B\rangle \langle \psi_B | U_{AB}^+ |\psi_B\rangle
\]

\[
= \sum_B \langle \psi_B | U_{AB} \rho_A \otimes |\psi_B\rangle \langle \psi_B | U_{AB}^+ |\psi_B\rangle
\]

\[
A_k \rightarrow A_k^+ \quad \text{ops on system-B}
\]

\[
\rho_A \rightarrow \sum A_k \rho_A
\]

\( A_k \) are "Kraus operators"

"Kraus representation"

"Operator sum representation" ("OSR")
Lecture 19

Unilocity demands

\[ \sum A_k^+ A_k = \sum_k \langle k | U_{AB}^+ | k \rangle \langle k | U_{AB} 14 \rangle \]

\[ = \langle \Phi | U_{AB}^+ U_{AB} 14 \rangle \Phi \]

\[ \sum A_k^+ A_k = 1 \]

More general evolution?

2. Initial state may be entangled:

- **Def:** A state \( \rho \) is entangled if it cannot be written

\[ \rho = \sum_i \lambda_i \rho_{A_i} \otimes \rho_{B_i} \], otherwise it is separable.

Example: \( \rho = \frac{1}{2} (1001 + 1111) (\frac{3}{5})(1001 + 1111) = \)

\[ = \frac{1}{2} (100X00 + 100X11 + 111X01 + 111X10) \]

\[ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \] is entangled.

- Could have conditional evolution via a measurement \( \mathcal{E} \)?

after \( U_{AB} \). (More on this later.)
General map: \( p \rightarrow p' = \mathcal{A}(p) \) is linear.

(Proof: Convexity)

2. Convex-linear: \( \mathcal{A}(\lambda p_1 + (1-\lambda) p_2) = \lambda \mathcal{A}(p_1) + (1-\lambda) \mathcal{A}(p_2) \) for all \( \lambda \).

(Preseves ensemble interpretation) \( \rightarrow \) connected to linear map

3. Hermitian-preservation: \( \rho = \rho^+ \Rightarrow \mathcal{A}(\rho) = \mathcal{A}(\rho)^+ \)

4. Trace-preserving: \( \text{Tr}(\rho) = 1 \Rightarrow \text{Tr}(\mathcal{A}(\rho)) = 1 \)

5. Positive ("positivity-preserving") \( \rho \geq 0 \Rightarrow \mathcal{A}(\rho) \geq 0 \) (non-negativity)

6. Completely positive: \( \mathcal{A} \otimes 1_b \) is positive for all systems \( b \).

Note: \( \odot \Rightarrow \odot \) and \( \odot \Rightarrow \odot \)

So, completely positive trace-preserving map (CPTP) map

Kraus Representation Theorem: Every CPTP map has an OSR

Proof: See Preskill's notes 8.3.3 or Thm 8.1 in NC

Ambiguity in the OSR:

\[ \mathcal{A}(\rho) = \sum_{k=1}^{m} A_k \rho A_k^+ \]

\[ \mathcal{B}(\rho) = \sum_{k=1}^{n} B_k \rho B_k^+ \]

\[ \mathcal{A} = \mathcal{B} \Rightarrow A_j = \sum_{i=1}^{m} u_{ij} B_k \]

where \( u_{ij} \) is an \( m \times n \) matrix

(approx. "zero" Kraus ops to smaller list)