Brad's Wiki Tutorial - Nights?

$\boxed{\text{Ph 452}}$ Lecture 19

QI quick review

States: $|\psi\rangle \longrightarrow \rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$

$\mathrm{tr}\,\rho = 1,\ \rho = \rho^\dagger,\ \rho \geq 0$

Dynamics:

$$\left.\begin{array}{l} |\psi\rangle \to U|\psi\rangle \\ \langle\psi| \to \langle\psi|U^\dagger \end{array}\right\} \Rightarrow \rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$\to \sum \lambda_i U|\psi_i\rangle\langle\psi_i|U^\dagger$$

$$= U\rho U^\dagger$$

Measurement:

$$|\psi\rangle \to \frac{\Pi_k|\psi\rangle}{\sqrt{\mathrm{prob}(k)}} \Rightarrow \rho = \sum \lambda_i |\psi_i\rangle\langle\psi_i|$$

$$\to \frac{\sum \lambda_i \Pi_k|\psi_i\rangle\langle\psi_i|\Pi_k}{\mathrm{prob}(k)}$$

But $\mathrm{prob}(k) = \mathrm{tr}(\Pi_k \rho)$, so

$$\rho \to \frac{\Pi_k \rho \Pi_k}{\mathrm{tr}(\Pi_k \rho)} \quad \text{w/ prob } \mathrm{tr}(\Pi_k \rho)$$ "Conditional evolution"

Discarding outcome label $\to$ get an ensemble average!

$$\rho \to \sum \cancel{\mathrm{prob}(k)} \frac{\Pi_k \rho \Pi_k}{\cancel{\mathrm{tr}(\Pi_k \rho)}} = \sum \Pi_k \rho \Pi_k$$

"Unconditional evolution"

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Open System dynamics

Recall $\rho_A = tr_B |\psi\rangle_{AB\,AB}\langle\psi|$.

Partial trace averages over outcomes of auxilliary system.

e.g. $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$tr_B |\psi\rangle_{AB\,AB}\langle\psi| = \frac{1}{2}\ {}_B\langle 0|(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)|0\rangle_B$$

$$+ \frac{1}{2}\ {}_B\langle 1|(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)|1\rangle_B$$

$$= \frac{1}{2}(|0\rangle_{A\,A}\langle 0| + |1\rangle_{A\,A}\langle 1|)$$

Quantum Operation:

CPTP map

Superop.

$$\rho_A \to tr_B(U_{AB}\,\rho_A \otimes \sigma_B\, U_{AB}^\dagger) \qquad \text{"Unitary representation"}$$

Wlog $\sigma_B = |\psi\rangle_{B\,B}\langle\psi|$ ("Church of the larger Hilbert space")

$$\rho_A \to \sum {}_B\langle k|\, U_{AB}\, \underset{\text{can swap}}{\rho_A \otimes |\psi\rangle_B}\, {}_B\langle\psi|\, U_{AB}^\dagger\, |k\rangle_B$$

$$= \sum \underbrace{{}_B\langle k|U_{AB}|\psi_B\rangle}_{A_k}\, \rho_A\, \underbrace{\langle\psi|U_{AB}^\dagger|k\rangle_B}_{A_k^\dagger}$$

ops on system A

$$\boxed{\rho_A \to \sum A_k \rho A_k^\dagger}$$

A_k are "Kraus operators"

"Kraus representation"

"Operator sum representation" ("OSR")

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Unitarity demands

$$\sum A_k^\dagger A_k = \sum_k {}_B\langle\psi| U_{AB}^\dagger |k\rangle\langle k|_{BB} U_{AB} |\psi\rangle$$

$$= {}_B\langle\psi| U_{AB}^\dagger U_{AB} |\psi\rangle_B$$

$$\boxed{\sum A_k^\dagger A_k = 1}$$

More general evolution?

(a) Initial state may be entangled:

Deciding is an NPC problem.

Def: A state ρ is entangled if it cannot be written as

$$\rho = \sum_i \lambda_i \rho_A^i \otimes \rho_B^i \text{ .} \quad \text{Otherwise it is separable.}$$

Example: $\rho = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)(\frac{1}{\sqrt{2}})(\langle 00| + \langle 11|) =$

$$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{is entangled}$$

(b) Could have conditional evolution via a measurement $\{\Pi_{AB}^i\}$ after U_{AB}^i. (More on this later.)

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General map $\rho \Rightarrow \rho' = \mathcal{A}(\rho)$ desiderata

"convex linear"

(a) Convex-linear: $\mathcal{A}(\lambda \rho_1 + (1-\lambda)\rho_2) = \lambda \mathcal{A}(\rho_1) + (1-\lambda)\mathcal{A}(\rho_2)$ $\quad 0 \le \lambda \le 1$

(Preserves ensemble interpretation) $\Rightarrow$ can extend to linear map

(b) Hermeticity-preserving: $\rho = \rho^\dagger \Rightarrow \rho' = \rho'^\dagger$

(c) Trace-preserving: $\mathrm{tr}\,\rho = 1 \Rightarrow \mathrm{tr}\,\rho' = 1$

Note: $\mathcal{A} \otimes I$ is trace-preserving + linear

(d) Positive ("positivity preserving") $\rho \geq 0 \Rightarrow \rho' \geq 0$ (non-neg eigenvalues)

(e) Completely positive: $\mathcal{A} \otimes I_B$ is positive for all systems B.

Transpose map: Not CP

Note (e) $\Rightarrow$ (d) and (d) $\Rightarrow$ (b)

So Completely Positive Trace-Preserving map (CPTP) map

Kraus Representation Theorem: Every CPTP map has an OSR

Proof: See Preskill's notes §3.3 or thm 8.1 in N+C

Skip: Jamiolkowski isomorphism

Ambiguity in the OSR:

$$\mathcal{A}(\rho) = \sum_{k=1}^{m} A_k \rho A_k^\dagger, \quad \mathcal{B}(\rho) = \sum_{k=1}^{n} B_k \rho B_k^\dagger$$

$$\mathcal{A} = \mathcal{B} \Rightarrow \boxed{A_j = \sum u_{jk} B_k}$$ where U_{jk} is an $m \times m$ matrix

(append "zero" Kraus ops to smaller list)

Got this far