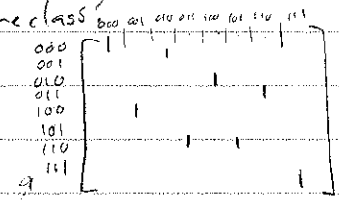
5/23/07

Ph 452 Lecture 2

"Question: Did anyone not receive my test e-mail to the class?"

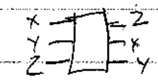



## IQI quick review:

**Bits + Gates:** $\mathbb{B}$, $f: \mathbb{B}^n \to \mathbb{B}^m$

③ · Reversible / irreversible gates (permutation: not its own inverse, but reversible)

② · Universality: all fs composable from {NAND, FANOUT} or {Tof}




① · Serial/Parallel: $f(g(\vec{x}))$ or $(f(\vec{x}), g(\vec{x}))$

**Pbits + Pgates:** probability vectors, stochastic matrices

② · Inherits gates + universality from bits + gates (0-1 vectors, matrices)

① · Serial/Parallel: $P_2 P_1 \vec{p}$ or $P_2 \otimes P_1 \vec{p}$

③ · Not all pbits factorizable: classical correlations

$$\vec{p} = \begin{array}{c} \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix} \end{array} \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array}$$

00 prob 1/2

11 prob 1/2

**Nonlinear pbit dynamics:** Incorporating new information

Bayes' Rule: $$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{P(B|A)P(A)}{\sum_i P(B|A_i)P(A_i)}$$

Solicit actual example from class

was "handedness"

Example: pregnancy tests

A = pregnant

B = test positive

$P(B|A) = 0.99$ (true positive rate)

$P(A) = 0.75$ (75% already sure)

$P(B|\bar{A}) = 0.10$ (false positive rate)

$P(\bar{A}) = 0.25$

$$P(A|B) = \frac{0.99(0.75)}{0.99(0.75) + 0.10(0.25)} = \boxed{0.97}$$

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Example: 2 pbits $\vec{p} = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix} \begin{matrix} 00 \\ 01 \\ 10 \\ 11 \end{matrix}$

New information: told the parity of the pbits (both same or both differ)

Q: What is the new $\vec{p}'$?

A:

If parity is ~~even~~ (both same):

$A =$ pbits are 00 $\quad P(A) = 1/4 \quad P(B|A) = 1$

$B =$ pbits are same $\quad P(B) = 1/2 \quad P(A|B) = \frac{1 \cdot \frac{1}{4}}{1/2} = 1/2$

$A =$ pbits are 01 $\quad P(A) = 1/4 \quad P(B|A) = 0$

$P(B) = 1/2 \quad P(A|B) = \frac{0 \cdot 1/4}{1/2} = 1/2$

$A =$ pbits are 11 $\quad P(A|B) = \frac{1 \cdot 1/4}{1/2} = 1/2$

$A =$ pbits are 10 $\quad P(B|A) = \frac{0 \cdot 1/4}{1/2} = 0$

$$\Rightarrow \vec{p}' = \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}$$

If parity is odd (both differ) $\vec{p}' = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix}$ (work out at home)

Compact notation:

$$\vec{p}' = \frac{\Pi_k \vec{p}}{\sum_i (\Pi_k p)_i}$$ for outcome k.

sum all elements of vector

$$\Pi_0 = \begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 1 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}$$

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Thm: General "measurement" dynamics for pbits:

$$\boxed{\vec{p} \to \frac{\Pi_k \vec{p}}{\sum_i (\Pi_k \vec{p})_i} \qquad \begin{array}{lll} \text{(a)} & \sum \Pi_k = I & \text{"complete set"} \\ \text{(b)} & \text{Each } \Pi_k \text{ is diagonal} & \text{diagonal} \\ \text{(c)} & \Pi_k \Pi_j = 0 \text{ if } j \neq k & \text{orthogonal} \\ \text{(d)} & \Pi_k^2 = \Pi_k & \text{projectors} \end{array}}$$

(c)+(d): $\Pi_k \Pi_j = \Pi_k \delta_{jk}$ , $\delta_{jk} = \begin{cases} 0 & j \neq k \\ 1 & j = k \end{cases}$

(b): $\Pi_k \circ I = \Pi_k$ , $A \circ B := \begin{pmatrix} a_{00} b_{00} & a_{01} b_{01} \\ a_{10} b_{10} & a_{11} b_{11} \end{pmatrix}$ "Hadamard Product"

Nonlinear: $\frac{\Pi_k (\vec{p} + \vec{q})}{\sum_i (\Pi_k (\vec{p} + \vec{q}))_i} \neq \frac{\Pi_k \vec{p}}{\sum_i (\Pi_k \vec{p})_i} + \frac{\Pi_k \vec{q}}{\sum_i (\Pi_k \vec{q})_i}$

Questions?
Yes: ① Pbit POVMs? (cover later)
② pbit "quantum eraser"? (cover later too)

## Logically impossible pgates

$\sqrt{NOT}$ , i.e. $P : P^2 = NOT = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Proof:

$$P^2 = \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} \begin{pmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{pmatrix} = \begin{pmatrix} P_{00}^2 + P_{01} P_{10} & P_{01} (P_{00} + P_{11}) \\ P_{10} (P_{00} + P_{11}) & P_{11}^2 + P_{10} P_{01} \end{pmatrix}$$

Con't $= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and obey $P_{ij} \geq 0 \ \forall i,j$.

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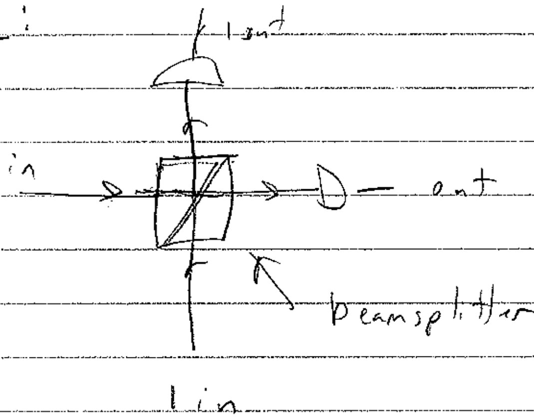
Quantum physics allows $\sqrt{NOT}$!

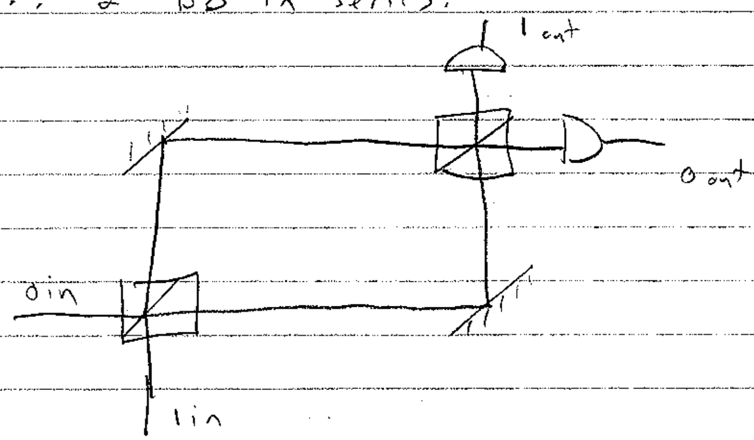
· **Beamsplitter**:




· Send 1 photon in either 0 in port or 1 in port
· Observe 1 photon out either 0 out port or 1 out port w/50% probability
· Photons are indivisible

$\Rightarrow$ This is a random switch $\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

But... 2 BS in series:




Expect: $\left(\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right)^2 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Observe: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ = NOT !!!

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Resolution: Probability axioms form a mathematical <u>theory</u> that doesn't describe <u>reality</u>.

What does? Quantum Mechanics!

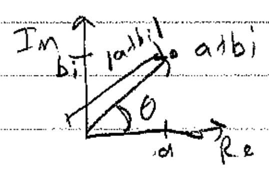
Key ideas: (a) Maximal information = probability amplitudes $\in \mathbb{C}$
$\neq$ complete information

(b) Bayes' Rule modified to Born Rule
- Allows more general measurements ($\Pi_k$ no longer diag)
- Conditional probabilities are $|\alpha|^2$'s, $\alpha \in \mathbb{C}$.

<u>Complex numbers:</u> $\mathbb{C} = \{ a+bi \mid a,b \in \mathbb{R},\ i^2 := -1 \}$

$$(a+bi)^* := a-bi$$

$$|a+bi|^2 := (a+bi)^*(a-bi) = a^2 - b^2$$




$$a+bi := |a+bi|(\cos\theta + i\sin\theta)$$

Euler formula: $e^{i\theta} = \cos\theta + i\sin\theta$

proof: use $e^x = 1 + x + \frac{x^2}{2!} + \cdots$

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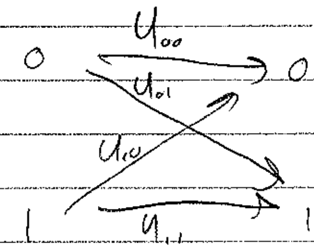
## Qubits & (qu)gates

qubits: $\vec{\psi} = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$ $\boxed{\begin{matrix} \psi_i \in \mathbb{C} \\ |\psi_0|^2 + |\psi_1|^2 = 1 \end{matrix}}$ $\boxed{\vec{\psi} \text{ and } e^{i\theta}\vec{\psi} \text{ represent same qubit state}}$

Gates: $U = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix}$




"transition (probability) amplitudes"

$$\boxed{\vec{\psi}' = U\vec{\psi}}$$

~~Got this far~~

## Dirac Notation: $\vec{\psi} \Leftrightarrow |\psi\rangle$

Why? Originally to handle $\infty$-dimensional vectors

$\vec{V}$ vs. $V_i$ $\qquad |\psi\rangle$ vs $\psi(x)$

$\rightarrow \psi(p)$ is $|\psi\rangle$ in a "new basis"

Also:

$$\vec{\psi}^{\dagger} = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}^{\dagger} = [\psi_0^* \ \psi_1^*] \Longleftrightarrow \langle\psi|$$

Why? Makes inner products pretty:

$$\|\vec{\psi}\|^2 = \vec{\psi}^{\dagger}\vec{\psi} = \underbrace{\langle\psi|}_{\text{bra}}\underbrace{|\psi\rangle}_{\text{ket}} \rightarrow \langle\psi|\psi\rangle$$

c

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Qubits : $|\psi\rangle$, $\langle\psi|\psi\rangle = 1$

Gates: $|\psi'\rangle = U|\psi\rangle$

Constraint: $|\psi'\rangle$ must be a valid qubit!

[1931] Wigner's Theorem: (essentially): $U$ must be a unitary matrix.

actually, could be antiunitary too.

$$\boxed{U^\dagger U = I}$$

("preserves unit length vectors")

Proof outline: $|\psi'\rangle = U|\psi\rangle$ and $\langle\psi'| = (|\psi'\rangle)^\dagger$

$$= (U|\psi\rangle)^\dagger$$
$$= (|\psi\rangle^\dagger U^\dagger)$$
$$= \langle\psi|U^\dagger$$

so $\langle\psi'|\psi'\rangle = (\langle\psi|U^\dagger)(U|\psi\rangle)$

$$\Rightarrow U^\dagger U = I$$

Example: Beamsplitter

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \qquad B^2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

so $B^2|\psi\rangle = \text{NOT}\,\underbrace{(i)|\psi\rangle}_{\text{same as } |\psi\rangle}$ (Can derive $B$ using QED.)

Serial/parallel composition : $U_2 U_1 |\psi\rangle$ and $U_1 \otimes U_2 |\psi\rangle$

Next time: Born rule, Quantum circuits.