Reminder: Wiki Tutorial 7pm tomorrow
Will e-mail topic suggestions — select topic by next Thu.
HW assignment — next two days

# Ph 45a Lecture 21

## IQI Quick Review

- Error channels: Bit-flip, phase-flip = phase damping

- Amplitude-damping channel

$$A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$$

Bloch sphere: $\frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$

$$\vec{r} \to \left(\sqrt{1-p}\, r_x,\ \sqrt{1-p}\, r_y,\ (1-p) r_z \underline{\underline{+p}}\right)$$

$$A_0 Z A_0^\dagger + A_1 Z A_1^\dagger = (1-p) Z, \text{ but also}$$

$$A_0 I A_0^\dagger + A_1 I A_1^\dagger = \begin{pmatrix} 1+p & 0 \\ 0 & 1-p \end{pmatrix} = I \underline{\underline{+pZ}} \quad \text{nonunital}$$

## Depolarizing channel:

$$\rho \to (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z) \qquad 0 \le p \le 1$$

Factoid: $\frac{1}{2}(I\rho I + X\rho X + Y\rho Y + Z\rho Z) = I$

Proof: Use $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$

$$\Rightarrow \rho \to (1-p)\rho + \frac{p}{3}(2I - \rho) = \left(1 - \frac{4}{3}p\right)\rho + \frac{4}{3}p\left(\frac{I}{2}\right)$$

$\Rightarrow$ with "probability" $\left(1 - \frac{4}{3}p\right)$ nothing happens, "prob" $\frac{4}{3}p$ $\rho \to I/2$.
$(p \le 3/4)$

(note: $I$ contains a component of $\rho$, so elements of "ensemble" are not independent.)

$\boxed{\text{Ph 45a}}$ Lecture 21

Bloch sphere picture

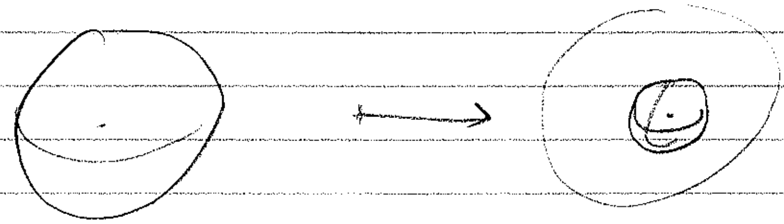
$$\tfrac{1}{2}(I+\vec{r}\cdot\vec{\sigma}) \rightarrow \left(1-\tfrac{4}{3}p\right)\left(\tfrac{1}{2}\right)(I+\vec{r}\cdot\vec{\sigma}) + \tfrac{4}{3}p\left(\tfrac{1}{2}\right)I$$

$$= \tfrac{1}{2}\left(I + \left(1-\tfrac{4}{3}p\right)\vec{r}\cdot\vec{\sigma}\right)$$

Q: If $p > 3/4$, does Bloch ball "invert"?

A: No. "can fit inside cigars"

$$\vec{r} \rightarrow \left(1-\tfrac{4}{3}p\right)\vec{r}$$



Bloch ball uniformly contracts.

Open Q. Systems

① $|\psi\rangle \rightarrow \rho$

② $U^\dagger U = I \rightarrow \sum A_i^\dagger A_i = I$  Aside: Davies thm: $d^2$ Kraus ops suffice

③ $\sum \Pi_j = I,\ \Pi_i \Pi_j = \delta_{ij}\Pi_j,\ \Pi_i = \Pi_i^\dagger \rightarrow$ ?

Q: How big does system B have to be?

A: $d_i^2$ (from Davies Thm?)

$$\rho_A \rightarrow \frac{1}{\text{prob}(k)} \mathrm{Tr}_B\left[\left(I_A\otimes\Pi_k^{(B)}\right) U_{AB}\left(\rho_A\otimes|\psi\rangle_B{}_B\langle\psi|\right)U_{AB}^\dagger\left(I_A\otimes\Pi_k^{\dagger(B)}\right)\right]$$

$$A_{jk} = {}_B\langle j|\,\Pi_k^{(B)}\,U_{AB}\,|\psi\rangle_B$$

$$\rho \rightarrow \sum_j A_{jk}\,\rho\,A_{jk}^\dagger / \text{prob}(k)$$

$$\sum_{jk} A_{jk}^\dagger A_{jk} = \sum_{jk}{}_B\langle\psi|U_{AB}^\dagger\,\Pi_k^{\dagger(B)}|j\rangle\langle j|\,\Pi_k^{(B)}U_{AB}|\psi\rangle_B$$

$$= I_A$$

$$\text{prob}(k) = \mathrm{tr}_{AB}\left[(I_A\otimes\Pi_k)\,U_{AB}\,(\rho_A\otimes|\psi\rangle\langle\psi|)\,U_{AB}^\dagger\,(I_A\otimes\Pi_k)\right]$$

$$= \mathrm{tr}_A\left(\sum_j A_{jk}^\dagger A_{jk}\,\rho_A\right)$$

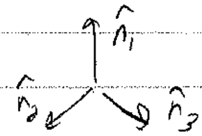
Ph 452 Lecture 21

Let $E_k := \sum_j A_{jk}^\dagger A_{jk}$

Then $\boxed{\underbrace{\sum_k E_k = I}_{\text{"measure"}}, \quad \mathrm{Prob}(k) = \mathrm{tr}(E_k \rho), \quad \underbrace{E_k \geq 0}_{\text{Positive operator}}}$

POVM: Positive operator valued measure.

Example: One-qubit POVM




$E_k = \frac{1}{3}(I + \hat{n}_k \cdot \vec{\sigma})$

$\sum E_k = 3\left(\frac{I}{3}\right) + \cancelto{0}{\left(\sum \hat{n}_k\right)} \cdot \vec{\sigma} = I, \quad E_k \geq 0$

Note: $\rho$ does change after a POVM, but the POVM formalism doesn't give a unique specification for the changes.

$\boxed{\text{Ph 452}}$ Lecture 21

POVM example:

Bob has $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = |+\rangle$. Which does he have?

$= \frac{1}{\sqrt{2}}(|\psi_1\rangle + |\psi_1^\perp\rangle)$

Any projective meas. will mis-identify sometimes

Proof:

Let $\{\Pi_k\}$ be a projective meas. where outcome $k$ implies Bob guesses "$|\psi_1\rangle$."

Then b/c $|\psi_2\rangle$ has a $|\psi_1\rangle$ component, Bob will sometimes guess "$|\psi_1\rangle$" when given $|\psi_2\rangle$

Now consider POVM:

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}}|1\rangle\langle 1|, \quad E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|-\rangle\langle -|)}{2}, \quad E_3 = I - E_1 - E_2$$

Q: What about $|0\rangle\langle 0|$, $|1\rangle\langle 1|$, which determines $|0\rangle$ some of the time, but never errs?

$\rightarrow$ Distinguishes $|\psi_1\rangle$ from $|\psi_2\rangle$ some of the time,

but <u>never</u> mis-identifies.

<u>Proof:</u> Given $|\psi_1\rangle$, $\text{prob}(1) = tr(E_1|\psi_1\rangle\langle\psi_1|) = 0$

$\Rightarrow$ if outcome "1" obtained, must have been $|\psi_2\rangle$

Given $|\psi_2\rangle$, $\text{prob}(2) = 0$ similarly

$\Rightarrow$ if outcome "2" obtained, must have been $|\psi_1\rangle$

If outcome "3" obtained $\rightarrow$ no decision.

Ph 452 Lecture 21

"Back to the main line of development: QEC"

"Generalized states, dynamics, measurements"

Quantum error: $|\psi_0\rangle\langle\psi_0| \rightarrow \sum A_i |\psi_0\rangle\langle\psi_0| A_i^\dagger = \rho_{final}$

How bad is it, doc?

Purity: $tr\,\rho^2 = \begin{cases} 1 & \rho = |\psi\rangle\langle\psi| \text{ for some } |\psi\rangle \quad (\text{recall } \Pi_i^2 = \Pi_i) \\ \leq 0 & \text{otherwise} \end{cases}$

Distance between qubits? $d(\rho,\sigma)$?

Step 1: distance between pbits? (pdits)

"Trace distance" or "Kolmogorov distance"

$D(\vec{p},\vec{q}) := \frac{1}{2}\sum_i |p_i - q_i|$

Features! $D(p,q) = D(q,p)$, $D(p,p)=0$, $D(p,q)+D(q,r) \geq D(p,r)$
$\rightarrow$ a "metric"

Motivation: maximum difference in probabilities of an event occuring:

$$D(\vec{p},\vec{q}) = \max_S |p(s) - q(s)| = \max_S \Big| \sum_{i\in S} p_i - \sum_{i \in S} q_i \Big|$$

Another measure: Fidelity:

$$F(\vec{p},\vec{q}) = \sum_i \sqrt{p_i q_i}$$

Features: Not a metric, $F(\vec{p},\vec{p}) = 1$, $F \geq 0$

Ph452 Lecture 21

For quantum states:

Trace distance: $D(\rho,\sigma) = \frac{1}{2}\,\mathrm{Tr}\,|\rho-\sigma|$

recall $|A| = \sqrt{A^\dagger A}$

For qubits:

$$D(\rho,\sigma) = \frac{1}{2}\mathrm{tr}\left|\frac{1}{2}(I+\vec{r}\cdot\vec{\sigma}) - \frac{1}{2}(I+\vec{s}\cdot\vec{\sigma})\right|$$

$$= \frac{1}{4}\,\mathrm{tr}\left|(\vec{r}-\vec{s})\cdot\vec{\sigma}\right|$$

$(\vec{r}-\vec{s})\cdot\vec{\sigma}$ has eigen vals $\pm\,|\vec{r}-\vec{s}|$, so

$$\boxed{D(\rho,\sigma) = \frac{1}{2}|\vec{r}-\vec{s}|}$$ $\frac{1}{2}$ Euclidean dist btw $\vec{r},\vec{s}$!

Got this far

Features: ① D is a metric, ② $D(U\rho U^\dagger, U\sigma U^\dagger) = D(\rho,\sigma)$

③ $D = \max_{\substack{\text{POVMs}\\ \{E_k\}}} \mathrm{tr}\left[E_k(\rho-\sigma)\right]$

"Max diff. in probabilities any POVM will give"

④ $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho,\sigma)$

"contractive under quantum operations"

⑤ Convex: $D\left(\sum p_i \rho_i, \sigma\right) \leq \sum p_i D(\rho_i,\sigma)$