Wiki, HW posti soon.

Errata: $F = \underline{\underline{Tr}}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$, not $\sqrt{tr\,\rho^{1/2}\sigma\rho^{1/2}}$

$E_k \neq U_k E_k U_k^\dagger$

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IQI Quick Review

- Depolarizing channel

$$\boxed{\rho = \left(1 - \tfrac{4}{3}p\right)\rho + \left(\tfrac{4}{3}p\right)\frac{I}{2} \qquad 0 \le p \le 1}$$

$\frac{I}{2}$ has a <u>component</u> of ρ in it, so ρ, $I/2$ are not <u>independent</u>.

$p > 3/4$ does <u>not</u> give an "inversion."

- POVM: $\boxed{\sum E_k = I, \quad E_k \ge 0, \quad prob(k) = tr(E_k\rho)}$

State after a POVM: wlog $E_k = A_k^\dagger A_k$, so

$$\boxed{\rho \to \frac{A_k \rho A_k^\dagger}{tr(A_k \rho A_k^\dagger)}}$$

[Not unique! $E_k = U_k^\dagger A_k^\dagger A_k U_k$
$E_k = \sum_j A_{jk}^\dagger A_{jk}$]

New today →

<u>Neumark's Dilation Theorem</u>: All POVMS can be "lifted" to a projective measurement on a larger space

Davies' thm

<u>Stinespring Dilation Theorem</u>: All quantum operations can be "lifted" to unitary dynamics on a larger space

- POVM Application: Unambiguous State Discrimination

<u>Input</u>: $|\psi_1\rangle$ or $|\psi_2\rangle$ w/50% $\qquad \langle\psi_1|\psi_2\rangle \neq 0$

<u>Object</u>: Maximize probability that measurement is conclusive

① $\{|\psi_1\rangle\langle\psi_1|, |\psi_1^\perp\rangle\langle\psi_1^\perp|\} \to p = \frac{1}{2}(1 - |\langle\psi_1|\psi_2\rangle|^2) = \frac{1}{2} - \frac{1}{2}\langle\psi_1|\psi_2\rangle\langle\psi_2|\psi_1\rangle$

② $\{|\psi_2\rangle\langle\psi_2|, |\psi_2^\perp\rangle\langle\psi_2^\perp|\} \to$ " " "

New twist today →

③ $E_1 = \frac{1 - |\psi_2\rangle\langle\psi_2|}{1 + |\langle\psi_1|\psi_2\rangle|}$, $E_2 = \frac{1 - |\psi_1\rangle\langle\psi_1|}{1 + |\langle\psi_1|\psi_2\rangle|}$, $E_3 = 1 - E_1 - E_2 \to p = 1 - |\langle\psi_1|\psi_2\rangle|$

(3) $E_1 = \dfrac{1 - |\psi_1\rangle\langle\psi_1|}{1 + |\langle\psi_1|\psi_2\rangle|}$ $\quad E_2 = \dfrac{1 - |\psi_2\rangle\langle\psi_2|}{1 + |\langle\psi_1|\psi_2\rangle|}$, $E_3 = 1 - E_1 - E_2$

$$\text{prob}(1 \,|\, |\psi_2\rangle) = \text{tr}(E_1 |\psi_2\rangle\langle\psi_2|) = \langle\psi_2|E_1|\psi_2\rangle$$

$$= \frac{1 - |\langle\psi_1|\psi_2\rangle|^2}{1 + |\langle\psi_1|\psi_2\rangle|} = 1 - |\langle\psi_1|\psi_2\rangle|$$

$$\text{Prob}(2 \,|\, |\psi_1\rangle) = 0$$

$$P(\text{conclusive}) = P(|\psi_1\rangle)\, P(2|\psi_1\rangle) + P(|\psi_2\rangle)\, P(1|\psi_2\rangle)$$
$$= \tfrac{1}{2}(2)(1 - |\langle\psi_1|\psi_2\rangle|$$

$$\boxed{P_{\text{concl.}} = 1 - |\langle\psi_1|\psi_2\rangle|}$$

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How similar are 2 pbits?

- $D(\vec{p},\vec{q}) = \frac{1}{2}\sum|p_i - q_i|$
- $F(\vec{p},\vec{q}) = \sum\sqrt{p_i q_i}$

How similar are 2 qubits?

- $D(\rho,\sigma) = \frac{1}{2}\mathrm{Tr}|\rho-\sigma|$
- New today → $F(\rho,\sigma) = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$

D Features:

1. $D(\rho,\sigma) = \frac{1}{2}|\vec{r}-\vec{s}|$ (qubits)
2. D is a metric
3. $D(U\rho U^\dagger, U\sigma U^\dagger) = D(\rho,\sigma)$
4. $D = \max\limits_{\substack{\text{POVMs}\\ \{E_k\}}} \mathrm{tr}[E_k(\rho-\sigma)]$ "max diff in probabilities any POVM will give"
5. $D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) \le D(\rho,\sigma)$ ("contractive")
6. $D(\sum p_i\rho_i,\sigma) \le \sum p_i D(\rho_i,\sigma)$ ("convex")

Aside: $D(\vec{p},\vec{q})$ features

2. D is a metric
3. $D(P\vec{p}, P\vec{q}) = D(\vec{p},\vec{q})$
4. $D = \max\limits_{S} \sum\limits_{i\in S} |p_i - q_i|$

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F features

① $F(\rho,\sigma) = F(\sigma,\rho)$

② $0 \le F(\rho,\sigma) \le 1.$ $F(\rho,\sigma) = 1 \iff \sigma = \rho$

(2b) $F(\rho,\sigma) = 0 \iff \rho, \sigma$ have support on orthogonal spaces.

③ $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho,\sigma)$

④ $F(\rho,\sigma) = \max_{\substack{|\psi\rangle:\ \rho = \mathrm{tr}_B|\psi\rangle \\ |\varphi\rangle:\ \sigma = \mathrm{tr}_B|\varphi\rangle}} |\langle\psi|\varphi\rangle|$ [Uhlmann's Thm]

⑤ $F(\rho,\sigma) = \min_{\mathrm{POVMs}\ \{E_k\}} F\left(\mathrm{tr}(\rho E_k), \mathrm{tr}(\sigma E_k)\right)$

⑥ $F(\rho, |\psi\rangle\langle\psi|) = \sqrt{\langle\psi|\rho|\psi\rangle}$, $F(|\varphi\rangle\langle\varphi|, |\psi\rangle\langle\psi|) = |\langle\psi|\varphi\rangle|$

⑦ $A(\rho,\sigma) = \arccos F(\rho,\sigma)$ is a metric

⑧ $F(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \geq F(\rho,\sigma)$

⑨ $F(\sum p_i \rho_i, \sigma) \geq \sum p_i F(\rho_i, \sigma)$ (F is concave)

Theorem:

$$\boxed{1 - F(\rho,\sigma) \le D(\rho,\sigma) \le \sqrt{1 - F(\rho,\sigma)}}$$

See NC § 9.2.3 for proof

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Quantum Error Correcting Criteria:

Noise channel $\mathcal{N}: \rho \to \sum A_k \rho A_k^\dagger$, $\sum A_k^\dagger A_k = I$

e.g. Bit-flip channel on 3 qubits:

$$A_0 = (1-p)^3\, I \otimes I \otimes I \qquad A_4 = (1-p)p^2\, 1XX$$
$$A_1 = (1-p)^2 p\, IIX \qquad A_5 = (1-p)p^2\, X1X$$
$$A_2 = (1-p)^2 p\, 1X1 \qquad A_6 = (1-p)p^2\, XX1$$
$$A_3 = (1-p)^2 p\, X11 \qquad A = p^3\, XXX$$

Correctible error channel $\mathcal{E}: \rho \to \sum E_k \rho E_k$, $\sum E_k^\dagger E_k \leq I$

e.g. $E_0 = A_0$ $\quad E_2 = A_2$
$\qquad E_1 = A_1$ $\quad E_3 = A_3$

$[[n, k, 2t+1]]$ code when $\{E_i\}$ = all errors of "Pauli weight" $\leq t$

e.g. Shor code is a $[[9,1,3]]$ code

Desire recovery operator $\mathcal{R}$ such that

$\propto$ b/c $\mathcal{E}$ is not TP

$$\mathcal{R} \circ \mathcal{E}(\rho_{code}) \propto \rho_{code}$$

e.g. ρ_{code} has basis $\{|\bar{0}\rangle = |000\rangle, |\bar{1}\rangle = |111\rangle\}$

Proof: Next time / Not Got This Far

Thm: $\mathcal{R}$ exists iff $\boxed{\langle \bar{i} | E_a^\dagger E_b | \bar{j} \rangle = C_{ab} \delta_{ij}}$, $C = C^\dagger$

Meaning:

① E_a on $|\bar{i}\rangle$ distinguishable from E_b on $|\bar{j}\rangle$: $\langle \bar{i} | E_a^\dagger E_b | \bar{j} \rangle = 0$ if $i \neq j$

② Measuring errors doesn't distinguish codewords: $\langle \bar{i} | E_a^\dagger E_b | \bar{i} \rangle = \langle \bar{j} | E_a^\dagger E_b | \bar{j} \rangle$