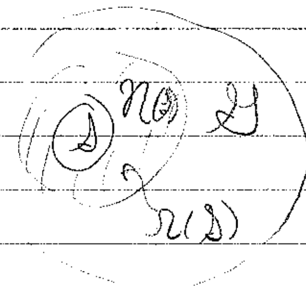
HW: due by 5pm
EC: due Thursday by 5pm.

Ph 452 Lecture 26

IQI Quick Review

- Pauli Group $\mathcal{G}_n$
- Stabilizer of a space $\mathcal{S}(\mathcal{C}) \leq \mathcal{G}_n$
- Codespace of a stabilizer $\mathcal{C}(\mathcal{S})$ $[\mathcal{C} \subseteq \mathcal{C}(\mathcal{S}(\mathcal{C}))]$ Explain → later
- Normalizer = centralizer $\mathcal{N}(\mathcal{S}) = \mathcal{Z}(\mathcal{S}) \geq \mathcal{S}$
- QEC and $\mathcal{C}(\mathcal{S})$:

    - $\Omega(\mathcal{S}) := \mathcal{N}(\mathcal{S}) \setminus \mathcal{S}$
        - "undetectable errors"
        - "encoded operations"



- $\mathcal{C}(\mathcal{S})$ corrects $\mathcal{E}$ if $\forall E_a, E_b \in \mathcal{E}$

$$\boxed{E_a^\dagger E_b \notin \Omega(\mathcal{S})}$$

- $$\boxed{\text{distance} = \min_{P \in \Omega(\mathcal{S})} wt(P)}$$

- degenerate iff $\exists E_a, E_b \in \mathcal{E}$ s.t. $E_a^\dagger E_b \in \mathcal{S}$

- Example: Bit-flip code

$\mathcal{G}_3$: $4^4 = 256$ elements

$S_1 = ZZI$ $\qquad \mathcal{S} = \langle S_1, \ldots, S_{n-k} \rangle$: $k=1$

$S_2 = IZZ$ $\qquad \mathcal{N} = \langle \bar{X}, \bar{Z} \rangle \times \mathcal{S} \times \langle i \rangle$ (64 elements)

$\bar{X} = XXX$ $\qquad \mathcal{E} = \{ III, XII, IXI, IIX \}$

$\bar{Z} = ZZZ$

$\mathcal{C}(\mathcal{S}) = \text{span}\{ |000\rangle, |111\rangle \}$

distance $= wt(ZII) = 1$

nondegenerate: No X_i, $X_i X_j \in \mathcal{S}$.

$[[3,1,1]]$ code

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How errors are detected

$$S_i|\psi\rangle = |\psi\rangle \qquad \forall |\psi\rangle \in C(\mathcal{S})$$

$S_i \in \mathcal{G}_n \Rightarrow S_i$ has ± 1 eigenvalues

→ $\boxed{\text{Measure } S_i \text{ observables!}}$

Any -1 eigenvalues → ERROR DETECTED!

Example: Bit-flip code

Measure $ZZ1$: $\Pi_+ = \frac{1}{2}(111 + ZZ1)$
$\Pi_- = \frac{1}{2}(111 - ZZ1)$

$$ZZ1 = (+1)\Pi_+ + (-1)\Pi_-$$

State: $|\psi\rangle = \alpha|101\rangle + \beta|000\rangle + \gamma|110\rangle$

$$|\psi\rangle \to \begin{cases} \Pi_+|\psi\rangle/\sqrt{p_+}, \text{ prob } p_+ = \langle\psi|\Pi_+|\psi\rangle & \text{"}ZZ1 = +1\text{"} \\ \Pi_-|\psi\rangle/\sqrt{p_-}, \text{ prob } p_- = \langle\psi|\Pi_-|\psi\rangle & \text{"}ZZ1 = -1\text{"} \end{cases}$$

$$\Pi_+|\psi\rangle = \frac{1}{2}\Big[\alpha\underbrace{(1 + (-1)(1))}_{0\,|101\rangle} + \beta\underbrace{(1+1)}_{|000\rangle} + \gamma\underbrace{(1 + (-1)(-1))}_{|110\rangle}\Big]$$
$$= \beta|000\rangle + \gamma|110\rangle$$

prob $|\beta|^2 + |\gamma|^2$

$\Pi_-|\psi\rangle = \alpha|101\rangle$, prob $|\alpha|^2$

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Measure $1ZZ$: $\tilde{\Pi}_\pm = \frac{1}{2}(111 \pm 1ZZ)$

$$\frac{1}{\sqrt{|\beta|^2+|\gamma|^2}}\left(\beta|000\rangle + \gamma|110\rangle\right) \to \begin{cases} \frac{\beta}{\sqrt{|\beta|^2+|\gamma|^2}}|000\rangle, & \text{prob } \frac{|\beta|^2}{|\beta|^2+|\gamma|^2} \quad \text{"}1ZZ=+1\text{"} \\ \frac{\gamma}{\sqrt{|\beta|^2+|\gamma|^2}}|110\rangle, & \text{prob } \frac{|\gamma|^2}{|\beta|^2+|\gamma|^2} \quad \text{"}1ZZ=-1\text{"} \end{cases}$$

phase irrelevant

$$\cancel{\frac{1}{\sqrt{|\alpha|^2}}\alpha}|101\rangle \to \begin{cases} 0, & \text{prob } 0 \quad \text{"}1ZZ=+1\text{"} \\ |101\rangle, & \text{prob} \quad \text{"}1ZZ=-1\text{"} \end{cases}$$

$ZZ1$, $1ZZ$ commute, so same outcomes if measured in any order

Summary

$ZZ1$	$1ZZ$	Post-measurement state		Recovered state	Probability
$+1$	$+1$	$\|000\rangle$	$\to$	$\|000\rangle$	$(\|\beta\|^2+\|\gamma\|^2)\left(\frac{\|\beta\|^2}{\|\beta\|^2+\|\gamma\|^2}\right) = \|\beta\|^2$
$+1$	-1	$\|110\rangle$	$\to$	$\|111\rangle$	$\|\gamma\|^2$
-1	$+1$	0			
-1	-1	$\|101\rangle$	$\to$	$\|111\rangle$	$\|\alpha\|^2$

QEC maps

$$|\psi\rangle \to \begin{cases} |000\rangle, & \text{prob } |\beta|^2 \\ |111\rangle, & \text{prob } |\alpha|^2+|\gamma|^2 \end{cases}$$

Can generalize to see how QEC works on a state $\rho = \sum_{\substack{ijk \\ lmn}} \alpha_{ijk}|ijk\rangle\langle lmn|$

Def: The (error) syndrome of a state ρ on $\mathcal{C}(\mathcal{S})$ is the spectrum of outcomes obtained by measuring a generating set of observables from $\mathcal{S}$ on ρ.

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Logical operators

For BFC, $\mathcal{N}(\mathcal{S}) = \langle \bar{X}, \bar{Z} \rangle \times \mathcal{S} \times \langle i \rangle$.

Generally an $[\![n,k]\!]$ code has

$$\mathcal{N}(\mathcal{S}) = \langle \bar{X}_1, \ldots, \bar{X}_k, \bar{Z}_1, \ldots, \bar{Z}_k \rangle \times \mathcal{S} \times \langle i \rangle$$

looks like $\mathcal{G}_k$.

Formally, define "quotient group" $\mathcal{L}(\mathcal{S}) := \text{"}\mathcal{N}(\mathcal{S})/\mathcal{S}\text{"} \cong \mathcal{G}_k$

$$\mathcal{N}(\mathcal{S}) = \mathcal{L}(\mathcal{S}) \times \mathcal{S}$$

What is $\mathcal{L}$?

$A \in \mathcal{L}(\mathcal{S}) \Rightarrow A = RS$ s.t. $S \in \mathcal{S}$, $R \notin \mathcal{S}$

Is $\mathcal{L}$ a group?

$$A, B \in \mathcal{L}(\mathcal{S}) \Rightarrow AB := (R_1 S_1)(R_2 S_2)$$
$$:= (R_1 R_2)(S_1 S_2) \quad (S_1 R_2 = R_2 S_1 \text{ b/c } R_2 \in \mathcal{N}(\mathcal{S}))$$
$$\in \mathcal{L}(\mathcal{S})$$

Leave it to you to verify identity, inverses, associative

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Examples

Shor code:

$$\mathcal{S}\left\{\begin{array}{l} S_1 = ZZ1 \quad 111 \quad 111 \\ S_2 = 1ZZ \quad 111 \quad 111 \\ S_3 = 111 \quad ZZ1 \quad 111 \\ S_4 = 111 \quad 1ZZ \quad 111 \\ S_5 = 111 \quad 111 \quad ZZ1 \\ S_6 = 111 \quad 111 \quad 1ZZ \\ S_7 = XXX \quad XXX \quad 111 \\ S_8 = 111 \quad XXX \quad XXX \end{array}\right.$$

$$\mathcal{L}\left\{\begin{array}{l} \bar{X} = ZZZ \quad ZZZ \quad ZZZ \\ \bar{Z} = XXX \quad XXX \quad XXX \end{array}\right. \qquad \text{or } \begin{array}{l} \bar{X} = XXX\ XXX\ XXX \\ \bar{Z} = ZZZ\ ZZZ\ ZZZ \end{array}$$

$n = 9, \quad k = 9-8 = 1$

$$|\bar{0}\rangle = \left(\tfrac{1}{\sqrt{2}}\right)^4 (|000\rangle + |111\rangle)^{\otimes 3} + \left(\tfrac{1}{\sqrt{2}}\right)^4 (|000\rangle - |111\rangle)^{\otimes 3}$$

$$\begin{aligned} d &= wt(\bar{X} S_7) \\ &= wt((XXX\ XXX\ XXX)(XXX\ XXX\ 111)) = \\ &= wt(111\ 111\ XXX) \\ &= 3 \\ &= wt(S_1 S_3 S_5 \bar{Z}) \\ &= wt(11Z\ 11Z\ 11Z) \end{aligned}$$

degenerate over $\mathcal{E} = \{P \mid wt(P) \leq 1\}$!

$$E_1 = Z_1, \quad E_2 = Z_2 \qquad E_1^{\dagger} E_2 = ZZ1\ 111\ 111 \in \mathcal{S}!$$

$\rightarrow$ $[\![9,1,3]\!]$ degenerate code.

$$\begin{aligned} \bar{X}|\bar{0}\rangle &= Z^{\otimes 9} \left(\tfrac{1}{\sqrt{2}}\right)^3 (|000\rangle + |111\rangle)^{\otimes 3} \\ &= \left(\tfrac{1}{\sqrt{2}}\right)^3 (|000\rangle - |111\rangle)^{\otimes 3} \\ &= |\bar{1}\rangle. \end{aligned}$$

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Example: 5-qubit code

$$S_1 = XZZX1 \qquad \text{Note } ZZX1X = S_1 S_2 S_3 S_4$$
$$S_2 = 1XZZX$$
$$S_3 = X1XZZ$$
$$S_4 = ZX1XZ$$
$$\bar{X} = XXXXX$$
$$\bar{Z} = ZZZZZ$$

$n = 5, \quad k = 5 - 1 = 1$

$d = wt(\bar{X} S_1) = wt(-1YY1X) = 3$

nondegenerate over $\mathcal{E} = \{ P \mid wt(P) \le 1 \}$

No $E_a^\dagger E_b \in \mathcal{S}$.

$\rightarrow$ $[\![5,1,3]\!]$ nondegenerate code

\# syndromes: $2^4 = 16$

\# errors: $\sum_{j=0}^{t} 3^j \binom{n}{j} = 3^0 \binom{5}{0} + 3^1 \binom{5}{1}$

$= 1 + 3(5) = 16$

Q: How does recovery work? Binary vector rep.

QHB saturated $\rightarrow$ perfect code.

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<u>Example:</u> 7-qubit Steane code

$$
\begin{aligned}
S_1 &= Z1Z1Z1Z \\
S_2 &= 1ZZ11ZZ \\
S_3 &= 111ZZZZ \\
S_4 &= X1X1X1X \\
S_5 &= 1XX11XX \\
S_6 &= 111XXXX \\
\hline
\bar{X} &= XXXXXXX \\
\bar{Z} &= ZZZZZZZ
\end{aligned}
$$

$[[7,1,3]]$ nondegen. code.

Mnemonic: All 3-bit #'s in binary in columns

$$
\begin{matrix}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{matrix}
$$

Example: $[[6,0,4]]$ code

$|0\rangle|\bar{0}\rangle + |1\rangle|\bar{1}\rangle$, where $|\bar{0}\rangle$, $|\bar{1}\rangle$ from 5-qubit code.