- Lecture 1, 2 on web
- |Q| #'s by e-mail
- PS later afternoon,

$\boxed{\text{Ph 458}}$ Lecture 3

1st PS on web, will be e-mailing you |Q| # for class

|Q| Quick review

- Bayes' Rule: nonlinear dynamics (normalization)
- $\sqrt{NOT}$: Not a p-gate, but done by beamsplitters → need QM!
- Complex number review
- Qubits: $\vec{\psi} = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}$ $|\psi_0|^2 + |\psi_1|^2 = 1$, $\vec{\psi} \Leftrightarrow e^{i\theta}\vec{\psi}$
- Gates: $U = \begin{bmatrix} U_{00} & U_{01} \\ U_{10} & U_{11} \end{bmatrix}$ $\vec{\psi}' = U\vec{\psi}$

Wigner's Thm (abridged) : $\vec{\psi}'$ is a valid qubit state iff U is unitary or antiunitary.

$$M^{+} := (M^{*})^{T} \quad \text{"Hermetian conjugate" or "dagger"}$$

$$\begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix}^{+} = \begin{bmatrix} M_{00}^{*} & M_{10}^{*} \\ M_{01}^{*} & M_{11} \end{bmatrix}, \quad \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix}^{+} = \begin{bmatrix} \psi_0^{*} & \psi_1^{*} \end{bmatrix}$$

$$M^{+}M = I \quad \text{(Unitary)}$$

$$M^{+}M = -I \quad \text{(Antiunitary)}$$

Thm: Quantum gates must be unitary.

Proof: See notes. Idea: infinitesimal gates; I is unitary.

Parallel: $\vec{\psi}_1 \otimes \vec{\psi}_2$, $U_1 \otimes U_2$

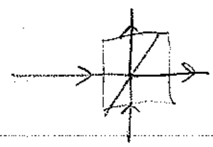
Serial: $U_2 U_1$

Ph 452 Lecture 3

Beamsplitter as $\sqrt{NOT}$

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

Why?



Quantum electrodynamics.

Roughly: In E&M, complex vectors for $\vec{E}$ used for bookkeeping of light waves' phase ϕ:

$$\vec{E}\, e^{i(kx-\omega t)} = \underbrace{\vec{E}\cos(kx-\omega t)}_{\text{physical part}} + i\vec{E}\sin(kx-\omega t)$$

In QED, complex $\vec{E}$ vector is light's probability amplitude.

π phase shift on hard reflection (Mirror, not BS)

$\rightarrow \pi/2$ relative phase shift on output ports of BS

$$B^2 = \frac{1}{2}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = NOT$$

(Could also calculate B using Fresnel coefficients r, t, with $|r|^2 + |t|^2 = 1$ and use energy conservation.)

Ph 43a Lecture 3

$\vec{\Psi}$: maximal info, not complete info

Incorporating new information: Born Rule

$$\boxed{\begin{array}{lll} \vec{\Psi} \to \dfrac{\Pi_k \vec{\Psi}}{\sqrt{\vec{\Psi}^\dagger \Pi_k \vec{\Psi}}} & (a)\ \sum_k \Pi_k = 1 & \text{"complete set"} \\ & (b)\ \Pi_k^\dagger = \Pi_k & \text{"Hermitian"} \\ \text{prob}(k|\vec{\Psi}) = \vec{\Psi}^\dagger \Pi_k \vec{\Psi} & (c)\ \Pi_k \Pi_j = \delta_{jk} \Pi_k & \text{"Orthogonal projection"} \end{array}}$$

Example 1: $\vec{\Psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\left\{ \Pi_0 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \Pi_1 = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}$

$$\text{Prob}(0|\vec{\Psi}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1/2$$

$$\vec{\Psi}_0 = \frac{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{1/2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Prob}(1|\vec{\Psi}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = 1/2$$

$$\vec{\Psi}_1 = \frac{\frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\sqrt{1/2}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Example 2: Non commuting measurements:

(a) $\vec{\Psi} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\left\{ \Pi_0' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \Pi_1' = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$\Psi \to \begin{cases} \vec{\Psi} & \text{Prob.} = 1 \\ 0 & \text{Prob} = 0 \end{cases}$$

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(b) $\vec{\psi} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$, $\{\Pi_0', \Pi_1'\}$

$$P(0|\vec{\psi}) = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1/2$$
$$P(1|\vec{\psi}) = \frac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1/2$$

$$\psi_0 = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{1/2}} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\psi_1 = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{\sqrt{1/2}} = \begin{bmatrix}0\\1\end{bmatrix}$$

(c) $\vec{\psi} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$, $\{\Pi_0', \Pi_1'\}$

$$P(0|\vec{\psi}) = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1/2$$
$$P(1|\vec{\psi}) = \frac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 1/2$$

$$\vec{\psi}_0 = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{1/2}} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\vec{\psi}_1 = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}{\sqrt{1/2}} = \begin{bmatrix}0\\1\end{bmatrix}$$

Synopsis:

Start in $\vec{\psi} = \begin{bmatrix}1\\0\end{bmatrix}$. Measure $\{\Pi_0, \Pi_1\}$. Measure $\{\Pi_0', \Pi_1'\}$.

$\rightarrow$ Obtain $\vec{\psi}$ again only with 50% chance!

Weird! Mathematical explanation: noncommuting measurements.

w.l.o.g., $\Pi_k' = U \Pi_k U^\dagger$

(Π_k's, Π_k''s not diagonal in same basis). Ignoring normalization, have

First Π_k', then Π_k (Example 1): $\vec{\psi} \rightarrow \Pi_j \Pi_k' \vec{\psi}$

First Π_k then Π_k' (Example 2): $\vec{\psi} \rightarrow \Pi_k' \Pi_j \vec{\psi}$

$\Pi_j \Pi_k' \neq \Pi_k' \Pi_j$, e.g. $\frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \neq \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$\frac{1}{2}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} \neq \frac{1}{2}\begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}$

$\boxed{\text{Ph 452}}$ Lecture 3

Dirac Notation

("bra") $|\psi\rangle := \vec{\psi} = \begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}$

("ket") $\langle\psi| := \vec{\psi}^\dagger \quad [\psi_0^* \ \psi_1^*]$

complex inner product: $\langle\psi|\varphi\rangle := \langle\vec{\psi}, \vec{\varphi}\rangle \Leftrightarrow \vec{\psi}\cdot\vec{\varphi} \Leftrightarrow \vec{\psi}^\dagger\varphi$

("bra(c)ket") $\quad [\psi_0^* \ \psi_1^*]\begin{bmatrix}\varphi_0 \\ \varphi_1\end{bmatrix} = \psi_0^*\varphi_0 + \psi_1^*\varphi_1$

outer product: $|\psi\rangle\langle\varphi| := |\psi\rangle\otimes\langle\varphi| \Leftrightarrow \vec{\psi}\,\varphi^\dagger$

$$\begin{bmatrix}\psi_0 \\ \psi_1\end{bmatrix}[\varphi_0^* \ \varphi_1^*] = \begin{bmatrix}\psi_0\varphi_0^* & \psi_0\varphi_1^* \\ \psi_1\varphi_0^* & \psi_1\varphi_1^*\end{bmatrix}$$

$$\left.\begin{aligned} |0\rangle &:= \begin{bmatrix}1 \\ 0\end{bmatrix} \\ |1\rangle &:= \begin{bmatrix}0 \\ 1\end{bmatrix}\end{aligned}\right\} \quad \{|0\rangle, |1\rangle\} \text{ are the computational basis.}$$

$|ab\rangle := |a\rangle\otimes|b\rangle$. E.g. $|110\rangle \Leftrightarrow |1\rangle\otimes|1\rangle\otimes|0\rangle$

$|i\rangle := |i\rangle$ in binary E.g. $|6\rangle \Leftrightarrow |\overset{4}{1}\overset{2}{1}\overset{1}{0}\rangle$

~~Got this far~~

Dirac notation++

$|A) := A$, a matrix $\begin{bmatrix}M_{00} & M_{01} \\ M_{10} & M_{11}\end{bmatrix}$

$(A| := A^\dagger$

$(A|B) := \operatorname{tr} A^\dagger B$ (Hilbert-Schmidt inner product)

$\operatorname{tr} M := \sum_i M_{ii}$

Q: Why Dirac notation?

A: Orig. to handle continuously-indexed vectors. $\quad \vec{V} : V_i :: |\psi\rangle : \psi(x)$

$V_i : \vec{e}_i\cdot\vec{V}. \quad \psi(x) = \langle x|\psi\rangle$