8/30/07

Ph 452 Lecture 4

|Q| Quick Review

Show $\langle v|u\rangle$

- Gates: unitary   Notes: Continuous time
- $B = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & i\\ i & 1\end{pmatrix} = \sqrt{NOT}$
- Born rule & noncommuting measurements
- Dirac notation

Q: Why Dirac Notation?

A: Continuously-indexed vectors, ("functionals")

	Discrete-indexed	Continuous-indexed	
	$\vec{v}$	$\|\psi\rangle$	
$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\hat{e}_i = [0\,0 \ldots \underbrace{0\,1\,0}_{i} \ldots 0]^T$	$\|x\rangle$	$\langle x\|x'\rangle = \delta(x-x')$
	$V_i = \hat{e}_i \cdot \vec{V} = \hat{e}_i^{\,T}\vec{V}$	$\psi(x) = \langle x\|\psi\rangle$	
	$\|\|\vec{V}\|\| = \sum \|V_i\|^2 = \vec{V}^T V$	$\|\|\,\|\psi\rangle\|\| = \langle\psi\|\psi\rangle$	

"So if Dirac notation was designed to handle continuously-indexed vectors, and we're only dealing with discretely-indexed ones in this class, why are we bothering to use it?

Well the answer is simple,

It's the same reason people climb Mt. Everest.

We do it because it's there."

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Discrete-time

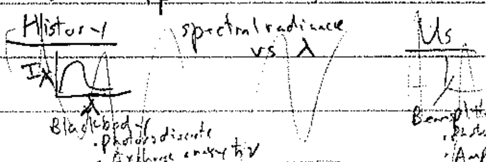
QM in Dirac Notation: $\mathcal{H}_2^{\otimes n}$ for "Hilbert Space"

$$\boxed{\begin{array}{l} \text{States: } |\psi\rangle \in \mathbb{C}_2^{\otimes n}, \quad \langle\psi|\psi\rangle = 1, \quad |\psi\rangle \cong e^{i\theta}|\psi\rangle \\ \text{Dynamics: } |\psi\rangle \to U|\psi\rangle, \quad U^\dagger U = I \\ \text{Measurement: } \text{prob}(k) = \langle\psi|\Pi_k|\psi\rangle, \quad |\psi\rangle \to \dfrac{\Pi_k|\psi\rangle}{\sqrt{p_k}} \\ \qquad \sum \Pi_k = I, \quad \Pi_k = \Pi_k^\dagger, \quad \Pi_k \Pi_j = \Pi_k \delta_{ij} \end{array}}$$

Q: Where's $\hbar$? A: Only shows up in continuous time extension!

(time has units; seconds)

Ideas put some physics in. Hallmark of an information theory: no units



Using Dirac Notation

Do Example 3 Now

Computational basis: $\{|i\rangle : i = 0, \ldots, 2^n - 1\}$ (aka "z-basis")

X-basis: $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

Matrices: $M = \sum_{ij} M_{ij} |i\rangle\langle j|$

$$M^\dagger = \sum M_{ji}^* |i\rangle\langle j| = \sum M_{ij}^* |j\rangle\langle i|$$

move here

Completeness:

$$\boxed{I = \sum |i\rangle\langle i|} \text{ for any basis}$$

Example 1: What is $\alpha|+\rangle + \beta|-\rangle$ in the z-basis?

Select basis.

A: $= \alpha I |+\rangle + \beta I |-\rangle$

$= \alpha \sum |i\rangle\langle i|+\rangle + \beta \sum |i\rangle\langle i|-\rangle = \sum \alpha(\langle i|+\rangle)|i\rangle + \sum \beta(\langle i|-\rangle)|i\rangle$

$\langle 0|+\rangle = \frac{1}{\sqrt{2}}[1\ 0]\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{\sqrt{2}}$ $\quad \langle 0|-\rangle = \frac{1}{\sqrt{2}}[1\ 0]\begin{bmatrix}1\\-1\end{bmatrix} = 1/\sqrt{2}$

$\langle 1|+\rangle = \frac{1}{\sqrt{2}}[0\ 1]\begin{bmatrix}1\\1\end{bmatrix} = 1/\sqrt{2}$ $\quad \langle 1|-\rangle = \frac{1}{\sqrt{2}}[0\ 1]\begin{bmatrix}1\\-1\end{bmatrix} = -1/\sqrt{2}$

$= (\alpha\frac{1}{\sqrt{2}} + \beta\frac{1}{\sqrt{2}})|0\rangle + (\alpha\frac{1}{\sqrt{2}} - \beta\frac{1}{\sqrt{2}})|1\rangle$

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Example 2: Prove $\mathrm{tr}\, AB = \mathrm{tr}\, BA$, where $\mathrm{tr}\, M := \sum M_{ii}$

Proof: $\mathrm{tr}\, AB = \sum \langle i|AB|i\rangle = \sum \langle i|A|j\rangle\langle j|B|i\rangle$
$= \sum \langle j|B|i\rangle\langle i|A|j\rangle = \sum \langle j|BA|j\rangle = \mathrm{tr}\, BA.$

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Example 3: Let $|\psi\rangle = \sum_{i,j=0}^{1} a_{ij}|i\rangle \otimes |j\rangle$.

Let A, B be 1-qubit gates.

Compute $A \otimes B|\psi\rangle$

Soln: $A \otimes B|\psi\rangle = \sum_{ij} a_{ij}\, A|i\rangle \otimes B|j\rangle$

Example 4: Projectors. $\Pi = \sum p_i |\phi_i\rangle\langle\phi_i|$, $\Pi^2 = \Pi$: $p_i^2 = p_i$: $p_i = 1, 0$

$$\mathrm{prob}(k) = \langle\psi|\Pi_k|\psi\rangle = \sum_i \langle\psi|\phi_i\rangle\langle\phi_i|\psi\rangle = \sum_i |\langle\psi|\phi_i\rangle|^2$$

Examples of Qubits

Linear algebra factoids

$M^\dagger M = MM^\dagger$: "Normal matrix" : Diagonalizable

diagonalizable w/ $|\psi\rangle\langle\psi|$ first! →

$$M = PDP^{-1}, \quad D = D \circ I = \mathrm{diag}(\vec{\lambda}) = \sum \lambda_i |\lambda_i\rangle\langle\lambda_i|$$
$$= \sum \lambda_i |\lambda_i\rangle\langle\lambda_i| \;:\; \lambda_i = \text{eigenvals},\ |\lambda_i\rangle = \text{eigenvectors},$$

$M = M^\dagger$: Hermitian, $\lambda_i \in \mathbb{R}$

$M^\dagger M = I$: Unitary : $\lambda_i = e^{i\theta_i}$

$f(M) = P f(D) P^{-1}$ e.g. $\sqrt{M^\dagger M} = P_{M^\dagger M} \sqrt{\;}$

$$= \sum \frac{f^{(n)}(0)\, M^n}{n!}$$

All matrices (even non-normal ones):

$M = U_M \sqrt{M^\dagger M}$: Polar decomposition, $U_M^\dagger U = I$

$M = U_M D V_M$: Singular value decomposition $U_M^\dagger U_M = V_M^\dagger V_M = I$

~~$\Pi^2 = \Pi$, $\Pi^\dagger$ ⇒ $\Pi = \sum|\phi_i\rangle\langle\phi_i|$: Hence $\mathrm{prob}(k) = \langle\psi|\Pi_k|\psi\rangle = \sum_i \langle\psi|\phi_i\rangle\langle\phi_i|\psi\rangle = \sum|\langle\psi|\phi_i\rangle|^2$~~

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Preskill 2.2

Physical examples of qubits: e^-

- $\boxed{\text{particle spin}}$ $\uparrow \downarrow$ $(e^-, \gamma, n, p, \ldots)$
- Two-Level Atoms $E=1$, $E=0$
- Current direction ↺ vs ↻ ("flux qubits")

Physical symmetries: Their action doesn't affect
- Measurement
- Dynamics

Particle symmetries: spacetime × internal

= spacetime × strong × weak × EM

$= \boxed{\text{Spin}(3,1)} \times SU(3) \times SU(2) \times U(1)$

Spacetime symmetries: 4-translations, 3-rotations, 3-Lorentz boosts

Little group: symmetries preserving linear momentum (p^μ)
- Massive particle: boost to rest frame → 3-rotations
- Massless particle: 2-rotations about $\hat{p}$.

Representing symmetry transformations

Measurement: $\text{prob}(k) = \sum |\langle \psi | \phi_i \rangle|^2$
- Must preserve $|\langle a|b\rangle|$ $\forall a,b$.
- Wigner's Thm, continuous symmetry ⇒ Unitary representation.
- Fundamental particles use irreducible representations:

$$\nexists M : \quad M\, U(g)\, M^{-1} = \begin{bmatrix} U_1(g) & \\ & U_2(g) \end{bmatrix} \quad \forall g \in \text{symmetry}.$$

Dynamics: $U_{sym} U_{gate} = U_{gate} U_{sym}$.

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spin labels which irrep of Little gp. particle transforms under.

e^-, n, p : spin $= 1/2$

$$U(\hat{n}, \theta) = e^{-i\theta \hat{n} \cdot (\frac{1}{2}\vec{\sigma})} = I \cos\frac{\theta}{2} - i \hat{n} \cdot \vec{\sigma} \sin\frac{\theta}{2}$$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$$

$$\sigma_x := X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_y := Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$ "minus i goes high"

$$\sigma_z := Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

(eigenvals of $\frac{1}{2}\sigma_i$ are $\pm 1/2$, hence "spin-$1/2$")

γ : spin $= 1$

$$U(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = e^{-i\theta \underbrace{\sigma_y}_{\text{eigenvals} = \pm 1}}$$

Actually, this is reducible [(block-) diagonalizable]

$$\begin{bmatrix} e^{i\omega/2} & \\ 0 & e^{-i\omega/2} \end{bmatrix}$$

eigvecs are $|R\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$, $|L\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

So really γ_R, γ_L are the fundamental particles

Got this far.