HW – front desk
changed #'s for 1st on Fri – sent e-mail to everyone
New HW, lecture notes posted later today

$\boxed{\text{Ph 452}}$ Lecture 5

1Q1 Quick Review

- Dirac notation examples (esp. completeness reln)
- Polar, singular-value decompositions
- Spin as a physical qubit

- Symmetry transforms (STs): physical, not informational

"So even though they are physical, they have consequences for information"

- Continuous STs (CSTs) must act like unitary gates
- "Little gp": nontranslational CSTs preserving $\vec{p}^2$, E

- Spin: tells how to represent LG CSTs by gates

spin-½: $U(\hat{n},\theta) = e^{-i\theta\,\hat{n}\cdot(\vec{\sigma}/2)}$ $\quad 0 \le \theta \le 4\pi$

spin-1: $U(\theta) = e^{-i\theta\sigma_y}$ $\quad 0 \le \theta \le 2\pi$

$\hat{n} = n_x\hat{x} + n_y\hat{y} + n_z\hat{z}$, $\quad n_x^2 + n_y^2 + n_z^2 = 1$

$$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \Leftrightarrow (X, Y, Z)$$

$$:= \underbrace{\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)}_{\text{Pauli matrices}}$$

$$\boxed{e^{-i\theta\,\hat{n}\cdot(\vec{\sigma}/2)} = I\cos\frac{\theta}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}}$$

The point: Meaning of "Rotate a qubit" depends on the physics of spin.

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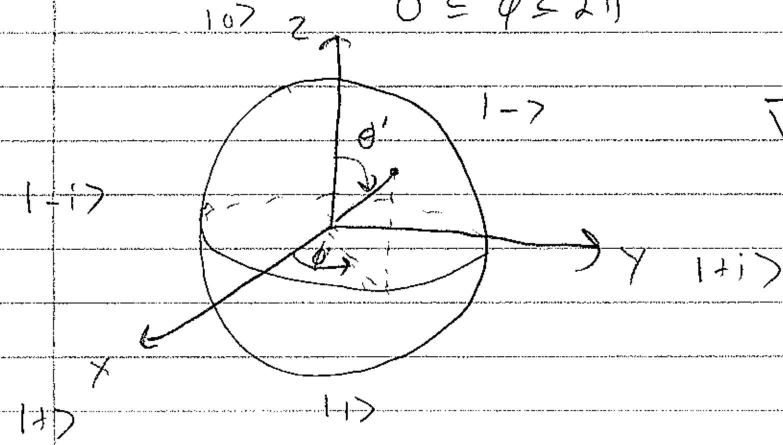
Visualizing spin rotations

General $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, $|\psi\rangle \cong e^{-i\theta}|\psi\rangle$

Wlog, $\boxed{|\psi\rangle = \cos\frac{\theta'}{2}|0\rangle + e^{i\phi'}\sin\frac{\theta'}{2}|1\rangle}$

$0 \le \theta' \le \pi$

$0 \le \phi \le 2\pi$



$\vec{V} = [\cos\phi'\sin\theta', \sin\phi'\sin\theta', \cos\theta']^T$

Spin-½: Bloch sphere, $\vec{V}$ = Bloch vector

Spin-1: Poincare sphere, $\vec{V}$ = Stokes vector

Feature: $\perp$ states are 180° apart

Example: spin-½

Spin-1	Spin-½	Qubit	Angles (θ', ϕ')
$\|R\rangle$	$\|\uparrow_z\rangle$	$\|0\rangle$	$(0, \text{undef})$
$\|L\rangle$	$\|\downarrow_z\rangle$	$\|1\rangle$	(π, undef)
$\|45\rangle$	$\|\uparrow_x\rangle$	$\|+\rangle := \frac{1}{\sqrt{2}}(\|0\rangle + \|1\rangle)$	$(\pi/2, 0)$
$\|-45\rangle$	$\|\downarrow_x\rangle$	$\|-\rangle := \frac{1}{\sqrt{2}}(\|0\rangle - \|1\rangle)$	$(\pi/2, \pi)$
$\|H\rangle$	$\|\uparrow_y\rangle$	$\|+i\rangle := \frac{1}{\sqrt{2}}(\|0\rangle + i\|1\rangle)$	$(\pi/2, \pi/2)$
$\|V\rangle$	$\|\downarrow_y\rangle$	$\|-i\rangle := \frac{1}{\sqrt{2}}(\|0\rangle - i\|1\rangle)$	$(\pi/2, -\pi/2)$

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Thm: Spin-½ $U(\hat{n},\theta)$ on $|\psi\rangle$ rotates $\vec{v}$ about $\hat{n}$ by θ.

Example:

$$U(\hat{z},\theta) = e^{-i\theta\hat{z}\cdot\vec{\sigma}/2} = e^{-i\theta/2\,\sigma_z} = I\cos\tfrac{\theta}{2} - iZ\sin\theta/2$$

$$= \begin{bmatrix} \cos\frac{\theta}{2} - i\sin\frac{\theta}{2} & \\ & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-i\theta/2} & \\ & e^{i\theta/2} \end{bmatrix}$$

$$|\psi\rangle = \begin{bmatrix} \cos\theta'/2 \\ e^{i\phi'}\sin\theta'/2 \end{bmatrix}$$

$$U|\psi\rangle = \begin{bmatrix} e^{-i\theta/2}\cos\theta'/2 \\ e^{i\theta/2+i\phi'}\sin\theta'/2 \end{bmatrix}$$

$$= e^{-i\theta/2}\begin{bmatrix} \cos\theta'/2 \\ e^{i(\phi'+\theta)}\sin\theta'/2 \end{bmatrix}$$

→ Net effect is $\phi' \mapsto \phi' + \theta$

Oddity: θ, $\theta + 2\pi$ give same U – why bother having

$0 \le \theta \le 4\pi$? (e.g. $U(\hat{n},2\pi) = -I \cong U(\hat{n},4\pi) = I$)

Ans: Physics! Controlled-U experiments show 2π, 4π spin CSTs are distinct

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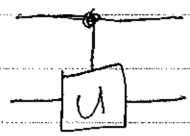

$$|0\rangle|\psi\rangle \rightarrow |0\rangle|\psi\rangle$$
$$|1\rangle|\psi\rangle \rightarrow |1\rangle U|\psi\rangle$$

$$|+\rangle|\psi\rangle \rightarrow |0\rangle|\psi\rangle - |1\rangle|\psi\rangle = |-\rangle|\psi\rangle$$

⟨ Example that physics can differentiate b/w 2π and 4π rotations even classically — Feynman's cup-in-hand demo ⟩

ignore. [~~Thm: Spin-1 $U(\theta)$ on $|\psi\rangle$ rotates $\vec{v}$ about $\hat{y}$ by 2θ~~]

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(David) Deutsch's Algorithm

Black-box functions: in $\boxed{f}$ out $\quad (y_0 \oplus f(x)_0, \dots, y_m \oplus f(x)_m)$

$\uparrow$ mod 2

Irreversible: $f : \mathbb{B}^n \to \mathbb{B}^m \qquad x \mapsto f(x)$

Reversible: " $\qquad (x, y) \mapsto (x, y \oplus f(x))$

Quantum: " $\qquad |x, y\rangle \mapsto |x, y \oplus f(x)\rangle$

aka "Oracle" $\quad X \in \underbrace{\mathbb{B}^m \times \cdots \times \mathbb{B}^m}_{n}$

Irreversible: $j \to X_j$

Reversible: $(j, k) \mapsto (j, k \oplus X_j)$

Quantum $\quad |j, k\rangle \mapsto |j, k \oplus X_j\rangle$

Quantum phase oracle: $\quad |j\rangle \to (-1)^{X_j} |j\rangle$

$$|j, -\rangle \mapsto |j\, X_j\rangle - |j\, \bar{X}_j\rangle = (-1)^{X_j} |j\rangle |-\rangle$$

$$\cancel{A^{(n)}|j,+\rangle = |j\,0\rangle + (-1)^{X_j}|j\,1\rangle}$$

Problem: Determine some property P of f by calling f on inputs.

$\Longleftrightarrow$

Evaluate a function g of X by querying components.

Challenge: Fewest f calls / X queries to evaluate P / g for worst-case f / X

Example: $X \in \mathbb{B}^2$, $g(X) = X_0 \wedge X_1$

$f : \mathbb{B} \to \mathbb{B} \quad P(f) = f(0) \wedge f(1)$

Best case: Query i, get $X_i = 0$. Then $X_0 \wedge X_1 = 0$!

Worst case: Query i, get $X_i = 1$. Need another query!

"Query complexity $= 2$"

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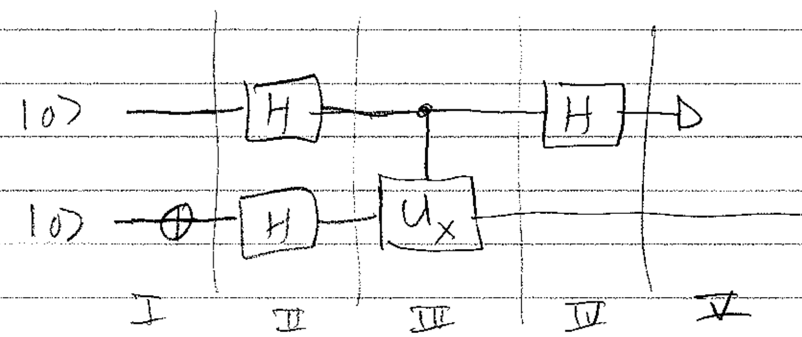
Example: $X \in \mathbb{B}^2$, $g(x) = X_0 \oplus X_1$

Best case: Same as worst case → always need 2 queries!

Deutsch's 1-query quantum algorithm:



(I)

$\oplus$: NOT gate: $|00\rangle \to |01\rangle$

(II)

$-\boxed{H}-$ $\quad H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ "Hadamard gate" (HW)

$$H|0\rangle = |+\rangle$$
$$H|1\rangle = |-\rangle$$

$$|01\rangle \to |+-\rangle$$

Got this far