No HW this week
HW due next Tue
1st make-up class: Tomorrow (Wed) 1-2:15, here (QECC)
HW returns remember lowest grade dropped
EC - goes to 58 max total

1.3 (c) Bayes rule + HW assignments
sd was wrong
See Brad for regrading.

$\boxed{\text{Ph453}}$ Lecture 9

|0⟩ Quick Review

D, R, Q relations for total fns

These only use H, CNOT → Deutsch's Algorithm (Related: Deutsch-Josza alg in NC, Simon's Alg in PS3)

To go beyond, want $U(\hat{n}, \theta)$'s, but ... Analog vs. Digital

→ Gates can only be approximated to some precision.

Q_1: How does imprecision scale with repetition?

$$\rightarrow E(U^n, V^n) \leq n\, E(U, V)$$

Q_2: Can finite gate sets approximate any U?

→ Universal gate bases "Quantum instruction sets"

Q_3: Can approximation be achieved efficiently?

→ Solovay-Kitaev algorithm "Quantum compiling"

Ph 452 Lecture 9

Approximation Error:

Put big space around $|v\rangle$

Vector norm: $\| \, |v\rangle \, \| := \sqrt{\langle v|v\rangle} \quad \left(= \| |v\rangle \|_2 \right)$

$\left(\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = v_x^2 + v_y^2 + v_z^2 \right)$

Operator norm: $\| M \| := \max\limits_{|v\rangle \neq 0} \dfrac{\| M |v\rangle \|}{\| \, |v\rangle \, \|}$

$$= \sqrt{\lambda_{max}(M^\dagger M)}$$

Properties:

(a) $\|X\| \geq 0$

(b) $\|X+Y\| \leq \|X\| + \|Y\|$ [Triangle Ineq.]

(c) $\|XY\| \leq \|X\| \, \|Y\|$

(d) $\|X^\dagger\| = \|X\|$

(e) $\|X \otimes Y\| = \|X\| \, \|Y\|$

(f) $\|U\| = 1$ if $U^\dagger U = I$

also $d(|u\rangle, |v\rangle) = \| |u\rangle - |v\rangle \|$

$$\boxed{E(U,V) := \|U - V\|}$$

Ph 452 Lecture 9

Thm: $E(U^n, V^n) \le n E(U,V)$

"Imprecision scales linearly"

Proof (sketch):

$$E(U^2, V^2) := \| UU - VV \|$$

$$= \| U(U-V) + (U-V)V \|$$

$$\le \|U\| \, \|U-V\| + \|U-V\| \, \|V\| \quad (b), (c)$$

$$= 2E(U,V) \quad \square \quad (f)$$

Universal gate sets ("Quantum instruction sets")

$(\forall U \mid U^\dagger U = I)(\forall \varepsilon > 0)(\exists \text{alg } V \text{ with gates from } \mathcal{G})[E(V,U) < \varepsilon]$

"Standard 3-bit basis"

$$\mathcal{G}_3 := \{H, S, S^{-1}, CNOT, TOF\}$$

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

"Shor basis"

$$\mathcal{G}_{Shor} := \{H, S, Tof\}$$

Let $X_\theta := U(\hat{x}, \theta)$, etc.

"Actual Shor basis":

$$\mathcal{G}'_{Shor} := \{Z_{\pi/2}, X_{\pi/2}, TOF\}$$

$\boxed{\text{Ph 45a}}$ Lecture 9

<u>Thm:</u> $\mathcal{G}_{Shor}$ and $\mathcal{G}_{Shor}'$ are <u>equivalent</u>

<u>Proof:</u>

$\Rightarrow \quad Z_{\pi/2} \cong S, \quad X_{\pi/2} \cong SHS$

$\Leftarrow \quad S \cong Z_{\pi/2}, \quad H \cong Z_{\pi/2} X_{\pi/2} Z_{\pi/2}$

<u>"Kitaev basis"</u>

$$\mathcal{G}_{Kit} := \{H, \Lambda(S)\}$$

Notation: $\Lambda(U)$ means $\left[\begin{array}{c|c} 1 & \\ & 1 \\ \hline & U \end{array}\right]$

e.g. $CNOT = \Lambda(X)$, $TOF = \Lambda(CNOT) = \Lambda^2(X)$.
$Z = \Lambda(-1)$, $S = \Lambda(i)$

<u>"Standard basis"</u>

$$\mathcal{G}_2 = \{H, S, T, CNOT\}$$

$$T := U(\hat{z}, \pi/8) = \begin{pmatrix} e^{-i\pi/8} & \\ & e^{i\pi/8} \end{pmatrix} \cong e^{-i\pi/8} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$
$$= \Lambda(e^{i\pi/4})$$

<u>"Shi CNOT basis"</u>

$$\mathcal{G}_{Shi} = \{B, CNOT \mid B^2 \neq B^2 \circ I, B \in \mathbb{R}^{2\times 2}\} \quad \text{"Basis-changing"}$$

e.g. $\{\sqrt{H}, CNOT\}$

$\boxed{\text{Ph 452}}$ Lecture 9

Aside:

Why "real" gates ($U \in \mathbb{R}^{n \times n}$) suffice:

→ Can simulate U_C on n qubits by U_R on n+1 qubits:

$$U_R |i\rangle|0\rangle := Re(U_c)\,|i\rangle|0\rangle + Im(U_c)|i\rangle|1\rangle$$

$$U_R |i\rangle|1\rangle := -Im(U_c)\,|i\rangle|0\rangle + Re(U_c)\,|i\rangle|1\rangle$$

$$U_c = Re(U_c) + i\,Im(U_c)$$

→ Don't need complex #'s for quantum mechanics!

Do need minus signs (negative probability amplitudes)

"Shi ToF basis"

$$\mathcal{G}'_{Shi} = \{B, Tof \mid B \neq B \circ I,\ B \in \mathbb{R}^{2 \times 2}\}$$

e.g. $\{H, Tof\}$

"Deutsch basis"

$$\mathcal{G}_{Deutsch} = \{\Lambda^2(i X_{\pi\alpha}) \mid \alpha \in \mathbb{R} \setminus \mathbb{Q}\}$$

Proof of universality of standard gate basis $\mathcal{G}_2$: See NC 4.5.3.

(Quantum Compiling)

Next time: Solovay-Kitaev alg, Grover alg.

Got this far