UNM Physics 452/581: Introduction to Quantum Information, Problem Set 2, Fall 2007

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Do all of the problems listed below. Hand in your problem set at the beginning of class on the desk at the front of the classroom or after class in the box in the Physics and Astronomy main office by 5 p.m. **Please put your name and/or IQI number number on your assignment**, as well as the course number (Physics 452/581). Please show all your work and write clearly. Credit will be awarded for clear explanations as much, if not more so, than numerical answers. Avoid the temptation to simply write down an equation and move symbols around or plug in numbers. Explain what you are doing, draw pictures, and check your results using common sense, limits, and/or dimensional analysis.

2.1. The Bourne Entanglement

On a quest to discover more about his mysterious origins, superspy Jason Bourne breaks into NSA headquarters to find answers. There he finds secret files indicating that the ultimate purposes of Operations Blackbriar and Treadstone that trained him to be an über-assassin were to protect the secrets of Operation Quantum. After engaging in many car chases and kung-fu fights across the globe, Bourne discovers the secret of Operation Quantum: the US government has prepared an enormous cache of entangled qubits, putting it light-years ahead of foreign governments who are just beginning to develop advanced quantum information technology. Each pair of entangled qubits the US government has are in the state

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Bourne picks up a pocketful of these entangled qubits from their secret hiding location and escapes—undetected, naturally.

(a) Bourne decides to test the entangled qubits he has. He decides to do a measurement that has four outcomes: $|++\rangle$, $|+-\rangle$, $|-+\rangle$, or $|--\rangle$. What is the probability that he will obtain each of these outcomes? (*Hint*: Use the Born rule, of course.)

(b) Bourne decides to express these states in a new basis—the eigenstates of the operator

$$\sigma_\theta := Z\cos\theta + X\sin\theta.$$

What are the eigenstates $|0_\theta\rangle$ and $|1_\theta\rangle$ of σ_θ? Use a labeling convention such that $|0_{\theta=0}\rangle = |0\rangle$ and $|1_{\theta=0}\rangle = |1\rangle$.

(c) Verify that no matter what θ Bourne chooses, the entangled qubits he has will look essentially the same. In other words, verify that for all θ, the following is true:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0_\theta 0_\theta\rangle + |1_\theta 1_\theta\rangle).$$

(d) Bourne decides to do some more testing on these qubits. This time, he does a measurement in which he measures the first qubit of the pair in the $\{|0\rangle, |1\rangle\}$ basis and the second qubit of the pair in the $\{|0_\theta\rangle, |1_\theta\rangle\}$ basis. Show that the probabilities of the various outcomes he obtains are given by

$$\begin{aligned}\mathrm{Prob}(|00_\theta\rangle) = \mathrm{Prob}(|11_\theta\rangle) &= \frac{1}{2}\cos^2\frac{\theta}{2}\\ \mathrm{Prob}(|01_\theta\rangle) = \mathrm{Prob}(|10_\theta\rangle) &= \frac{1}{2}\sin^2\frac{\theta}{2}.\end{aligned}$$

2.2. You don't know Jacques!

The Hadamard gate, named for mathematician Jacques Hadamard (1865–1963), is frequently employed in quantum algorithms. In the computational basis, it is the matrix

$$H := \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}.$$

In this problem, we explore this gate from many different vantage points.

(a) Calculate $H^\dagger$, $H^\dagger H$, H^2, $\det H$, and $\operatorname{tr} H$.

(b) Calculate the eigenvalues and eigenvectors of H.

(c) Express H as a linear combination of Pauli matrices.

(d) For each Pauli matrix A, report the matrix B such that $HA = BH$.

(e) Calculate $U(\hat{n}, \pi)$, where $\hat{n} = (\hat{x} + \hat{z})/\sqrt{2}$. How does this compare to H?

(f) Calculate $U(\hat{z}, \pi/2)U(\hat{x}, \pi/2)U(\hat{z}, \pi/2)$. How does this matrix compare to H?

(g) The *Walsh-Hadamard transform* is the n-bit gate

$$U_{WH} := \frac{1}{2^{n/2}}\sum_{j,k=0}^{2^n-1}(-1)^{j\cdot k}|j\rangle\langle k|,$$

where $j \cdot k$ denotes the bitwise inner product

$$(j_0, \ldots, j_{n-1}) \cdot (k_0, \ldots, k_{n-1}) := j_0k_0 \oplus \cdots \oplus j_{n-1}k_{n-1}$$

Show that $U_{WH} = H^{\otimes n}$. This is called a *product representation* of the transform.

(h) Let $\mathit{\Pi} := 2|0\rangle\langle 0| - I$, where I denotes the identity on n qubits and $|0\rangle$ denotes the n-qubit state $|00\ldots 0\rangle$. Show that

$$H^{\otimes n}\mathit{\Pi} H^{\otimes n} = 2|s\rangle\langle s| - I,$$

where $|s\rangle$ is defined as

$$|s\rangle := \frac{1}{2^{n/2}}\sum_{j=0}^{2^n-1}|j\rangle.$$

(i) Let X_a and Z_a denote the n-qubit gates defined as

$$X_a := \sum_{j=0}^{2^n-1} |j \oplus a\rangle\langle j|, \qquad\qquad Z_a := \sum_{j=0}^{2^n-1} (-1)^{j\cdot a}|j\rangle\langle j|,$$

where $j \cdot a$ denotes the bitwise inner product

$$(j_0, \ldots, j_{n-1}) \cdot (a_0, \ldots, a_{n-1}) := j_0 a_0 \oplus \cdots \oplus j_{n-1}a_{n-1}$$

and $j \oplus a$ denotes bitwise addition (modulo 2)

$$(j_0, \ldots, j_{n-1}) \oplus (a_0, \ldots, a_{n-1}) := (j_0 \oplus a_0, \ldots j_{n-1} \oplus a_{n-1}).$$

Show that $H^{\otimes n} X_a H^{\otimes n} = Z_a$. (*Hint*: Use the Walsh-Hadamard representation of $H^{\otimes n}$ from part (g), use the Kronecker delta δ_{ij} to collapse sums, and use the property that the bitwise dot product distributes over bitwise addition, *i.e.*, that $a \cdot (b \oplus c) = (a \cdot b) \oplus (a \cdot c)$.)

2.3. The Price is Right: *Royal Match*

In an attempt to boost ratings, Drew Carey, new host of the daytime game show *The Price is Right* has added a new game in which not one but *two* contestants have the chance to win some money. The game is called *Royal Match*, and it works as follows. Each contestant is given a card that says either "Match" or "No Match" on it. The audience gets to see what both cards say, and each contestant can see what his or her card says, but each contestant cannot see what the other contestant's card says. Each contestant then has to choose one of two cards he or she is going to display. One card is the King of Hearts, and the other card is the Queen of Hearts. If both of their secret cards say "Match" and they reveal a Royal Pair (King-Queen or Queen-King), then they each win \$1,000. If one of the secret cards says "No Match" and they don't reveal a Royal Pair (*i.e.*, they reveal King-King or Queen-Queen), then they also each win \$1,000. These are the only ways they can win; any other combination yields no reward.

Statisticians for *The Price is Right* have determined that the best *Royal Match* strategy will allow the pair of contestants to win 75% of the time. The space of strategies the statisticians considered allows for maximal collusion between the contestants, including prearranging their strategy and sharing correlated bits and pbits, but not cheating by informing the other contestant of the secret card's value. The statisticians used this expected winning percentage to set the game's cash value—advertising revenue allows the show to spend an average of \$1,500 per *Royal Match* game and still be profitable.

(a) Describe a strategy that allows the contestants to win *Royal Match* 75% of the time.

Years go by and *Royal Match* turns out to be a big hit. Curiously, the accountants at *The Price is Right* discover that they are paying out an average of \$1,707 per game. Although this might be a statistical fluke, this game has been played so many times that the accountants suspect that something else is up. They hire you as a consultant to dig up the truth. Reaching back to the wisdom you gained from your *Introduction to Quantum Information* class, you suspect that the parties are using entangled quantum states somehow.

After some tedious calculations, you arrive at the following possible strategy that the contestants could be using. The two parties, call them Alice and Bob for definiteness, prior to the game share an entangled qubit state between them of the following kind[1]:

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

To be concrete, suppose these are photonic qubits, having spin $s = 1$. (This concreteness allows an interpretation of the phrase "rotate a qubit by an angle such-and-such" below.) The strategy they undertake is as follows:

- If Alice receives "No Match," she measures her qubit and if she gets a 0 she reveals a King and if she gets a 1 she reveals a Queen.
- If Alice receives "Match," she rotates her qubit by $-\pi/8$, then measures, and again if she gets a 0 she reveals a King and if she gets a 1 she reveals a Queen.
- If Bob receives "No Match," he measures his qubit and if he gets a 0 he reveals a King and if he gets a 1 he reveals a Queen.
- If Bob receives "Match," he rotates his qubit by $\pi/8$, then measures, and again if he gets a 0 he reveals a King and if he gets a 1 he reveals a Queen.

(b) For each possible combination of "Match" and "No Match" cards that could be given to Alice and Bob, calculate the probability that their strategy yields a win.

(c) Assuming that each possible assignment of "Match" and "No Match" cards is equiprobable, calculate the probability that that this strategy is successful. Show that it equals roughly 80.18%, but give an exact analytic expression for the success probability.

Your analysis in this problem shows that quantum mechanics allows stronger cooperative behavior than is possible classically. The two contestants can win with a higher success probability than classical collusion allows, *even though they are not communicating with each other*. For this reason, quantum mechanics is said to possess "stronger" correlations than classical physics allows.

The careful reader will notice that 80.18% of \$2,000 is only \$1,604. So it seems that the parties are colluding even more cleverly. How are they doing this? Continue on to the extra credit problem if you want to find out more...

Extra Credit Problem

2.4 *Royal Match*, Take II

After learning of your results from problem 2.3, the show's statisticians want to readjust the prize amounts so that once again an average of \$1,500 per *Royal Match* game is awarded.

[1] Perhaps they got them from Jason Bourne. See problem 2.1.

(What a bunch of cheapskates!) The statisticians want to be sure that there isn't some better quantum protocol than the one you told them that yields an even higher success probability—they are pretty suspicious that there is one given the gap between the average \$1,604 payoff your strategy gives vs. the average \$1,707 they are paying out per game. You are hired again as a consultant to determine what the best possible quantum strategy for this game is.

You come up with the following:

- If Alice receives "No Match," she measures her qubit and if she gets a 0 she reveals a King and if she gets a 1 she reveals a Queen.
- If Alice receives "Match," she rotates her qubit by $\pi/4$, then measures, and again if she gets a 0 she reveals a King and if she gets a 1 she reveals a Queen.
- If Bob receives "No Match," he rotates his qubit by $\pi/8$, then measures his qubit, and if he gets a 0 he reveals a King and if he gets a 1 he reveals a Queen.
- If Bob receives "Match," he rotates his qubit by $-\pi/8$, then measures, and again if he gets a 0 he reveals a King and if he gets a 1 he reveals a Queen.

(a) For each possible combination of "Match" and "No Match" cards that could be given to Alice and Bob, calculate the probability that their strategy yields a win.

(b) Assuming that each possible assignment of "Match" and "No Match" cards is equiprobable, calculate the probability that that this strategy is successful. Show that it equals roughly 85.35%, but give an exact analytic expression for the success probability. (Note that 85.35% of \$2,000 yields the \$1,707 figure the show's accountants are observing—the show's contestants are apparently quite clever.)

It turns out that this is the optimal strategy for this game. Convincing the show's statisticians that there is no better quantum strategy is a task even beyond the scope of an extra credit problem for this class. (Although I seriously considered it.) The usual argument involves proving something called Tsirel'son's (or Cirel'son's) Inequality, which requires knowledge of quantum observable theory and sup-norm theory, neither of which we have discussed. A proof that the most general classical strategy can't win more than 75% of the time as shown in part (a) involves proving something called Bell's inequality, which is a bit easier to do, but also omitted as an explicit problem. Quantum strategies that win this game more than 75% of the time, such as those described in problems 2.3 and 2.4 are called "Bell inequality violating" in the quantum mechanics literature.

The Wikipedia entry for Tsirel'son's inequality is currently in awful shape. For those interested in learning more about this subject, radically improving this entry would make an excellent Wikipedia project.