UNM Physics 452/581: Introduction to Quantum Information, Solution Set 2, Fall 2007

2.1 The Bourne Entanglement

- (a) Let $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Let Π_i represent a projector onto of the four outcomes. From the Born rule, we have that the probability of obtaining the outcome i as $p_i = \langle\psi|\Pi_i|\psi\rangle$. We can write further write $\Pi_i = |i\rangle\langle i|$, where $|i\rangle \in \{|++\rangle, |+-\rangle, |-+\rangle, |--\rangle\}$, so that $p_i = |\langle\psi|i\rangle|^2$. We first find the representation of the Π_i in terms of the standard basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$:

$$|++\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

$$|+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)$$

$$|-+\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)$$

$$|--\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)$$

1. $$p_{++} = |\langle\psi|++\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)|^2 = \frac{1}{2}$$

2. $$p_{+-} = |\langle\psi|+-\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{2}(|00\rangle - |01\rangle + |10\rangle - |11\rangle)|^2 = 0$$

3. $$p_{-+} = |\langle\psi|-+\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{2}(|00\rangle + |01\rangle - |10\rangle - |11\rangle)|^2 = 0$$

4. $$p_{--} = |\langle\psi|--\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)|^2 = \frac{1}{2}$$

- (b) In order to find the eigenstates, we work with the explicit matrix form of σ_θ:

$$\sigma_\theta = Z\cos\theta + X\sin\theta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

The characteristic polynomial of this equation is

$$(\cos\theta - \lambda)(\cos\theta + \lambda) + \sin\theta^2 = \sin\theta^2 + \cos\theta^2 + \lambda^2 = 0$$

So our eigenvalues are $\lambda_{+1} = +1, \lambda_{-1} = -1$. We now find the associated eigenvectors by looking at the nullspace of $\sigma_\theta - \lambda_i I$. First for λ_{+1}:

$$\sigma_\theta - I = \begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix}$$

The unnormalized eigenvector is then $[1, \frac{1-\cos\theta}{\sin\theta}]^T$. For λ_{-1}, we have:

$$\sigma_\theta + I = \begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix}$$

The unnormalized eigenvector is then $[1, -\frac{1-\cos\theta}{\sin\theta}]^T$. Normalizing and making use of some half-angle formulas gives

$$|0_\theta\rangle = [\cos(\theta/2),\ \sin(\theta/2)]^T = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$$

$$|1_\theta\rangle = [-\sin(\theta/2),\ \cos(\theta/2)]^T = -\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle$$

You can directly verify that for $\theta = 0$ these simplify to $|0\rangle, |1\rangle$.

- (c) We proceed by directly using the derivations

$$\begin{aligned}
\frac{1}{\sqrt{2}}(|0_\theta 0_\theta\rangle + |1_\theta 1_\theta\rangle) &= \frac{1}{\sqrt{2}}[(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle)(\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle) \\
&+ (-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle)(-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle)] \\
&= \frac{1}{\sqrt{2}}[\cos^2\frac{\theta}{2}|00\rangle + \frac{\sin\theta}{2}(|01\rangle + |10\rangle) + \sin^2\frac{\theta}{2}|11\rangle \\
&+ \sin^2\frac{\theta}{2}|00\rangle - \frac{\sin\theta}{2}(|01\rangle + |10\rangle) + \cos^2\frac{\theta}{2}|11\rangle] \\
&= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \text{ for all } \theta
\end{aligned}$$

- (d) We proceed in an analogous manner to part (a).

$$|00_\theta\rangle = |0\rangle \otimes (\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle) = \cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}|01\rangle$$

$$|01_\theta\rangle = |0\rangle \otimes (-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle) = -\sin\frac{\theta}{2}|00\rangle + \cos\frac{\theta}{2}|01\rangle$$

$$|10_\theta\rangle = |1\rangle \otimes (\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle) = \cos\frac{\theta}{2}|10\rangle + \sin\frac{\theta}{2}|11\rangle$$

$$|11_\theta\rangle = |1\rangle \otimes (-\sin\frac{\theta}{2}|0\rangle + \cos\frac{\theta}{2}|1\rangle) = -\sin\frac{\theta}{2}|10\rangle + \cos\frac{\theta}{2}|11\rangle$$

Therefore

1.
$$p_{00_\theta} = |\langle\psi|00_\theta\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)(\cos\frac{\theta}{2}|00\rangle + \sin\frac{\theta}{2}|01\rangle)|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

2.
$$p_{01_\theta} = |\langle\psi|01_\theta\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)(-\sin\frac{\theta}{2}|00\rangle + \cos\frac{\theta}{2}|01\rangle)|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

3.
$$p_{10_\theta} = |\langle\psi|10_\theta\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)(\cos\frac{\theta}{2}|10\rangle + \sin\frac{\theta}{2}|11\rangle)|^2 = \frac{1}{2}\sin^2\frac{\theta}{2}$$

4.
$$p_{11_\theta} = |\langle\psi|11_\theta\rangle|^2 = |\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)(-\sin\frac{\theta}{2}|10\rangle + \cos\frac{\theta}{2}|11\rangle)|^2 = \frac{1}{2}\cos^2\frac{\theta}{2}$$

2.2 You don't know Jacques!

For reference,
$$H := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

- (a) Since H is real and symmetric, $H^\dagger = H$. Thefore $H^\dagger H = H^2 =$
$$\frac{1}{2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = I_2$$
$\det H = \frac{1}{2}(1\times(-1) - 1\times 1) = -1$ and $\mathrm{tr}\ H = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 0$.

- (b) We see that $H = \sigma_{\pi/4}$, where σ_θ was the matrix we worked with in question 1 part (b). The eigenvalues are ± 1, with eigenvectors $v_{+1} = [\cos\frac{\pi}{8},\ \sin\frac{\pi}{8}]^T$ and, $v_{-1} = [-\sin\frac{\pi}{8}, \cos\frac{\pi}{8}]^T$.

- (c) By inspection, we see
$$H = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right] = \frac{1}{\sqrt{2}}\left[Z + X\right]$$

- (d) The forms of Z and X are given in part (c) and
$$Y := \begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$$
For part (a) we know $H^2 = I$, so that $HAH == B$. Simply calculating, we find $HXH = Z$, $HZH = X$ and $HYH = -Y$.

- (e) Recall $U(\hat{n},\theta) = e^{-i\theta\hat{n}\cdot\vec{\sigma}/2} = I\cos\frac{\theta}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\theta}{2}$. For $\hat{n} = (\hat{x}+\hat{z})/\sqrt{2}$ and $\theta = \pi$, we have
$$U(\frac{\hat{x}+\hat{y}}{\sqrt{2}},\pi) = I\cos\frac{\pi}{2} - i(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\cdot(X,Y,Z)\sin\frac{\pi}{2} = -i\frac{1}{\sqrt{2}}(X+Z) = -iH$$
So up to a global phase, this is the same expression as we calculated in (c).

- (f)

$$\begin{aligned}
U(\hat{z},\pi/2)U(\hat{x},\pi/2)U(\hat{z},\pi/2) &= \frac{1}{\sqrt{2}}(I-iZ)\frac{1}{\sqrt{2}}(I-iX)\frac{1}{\sqrt{2}}(I-iZ) \\
&= \frac{1}{2\sqrt{2}}(I-iZ)(I-iX-iZ+iY) \\
&= \frac{1}{2\sqrt{2}}(I-iX-iZ+iY-iZ-iY-I-iX) \\
&= -i\frac{1}{\sqrt{2}}(X+Z) = -iH
\end{aligned}$$

Again, we have our original H up to a global phase.

- (g) Lets first start by rewriting the single-qubit H in terms of the basis matrix elements.

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1|\right) = \frac{1}{\sqrt{2}}\sum_{k,j=0}^{1}(-1)^{jk}|k\rangle\langle j|$$

This matches U_{WH} for $n=1$. We now generalize for N qubits

$$\begin{aligned}
H^{\otimes n} &= \bigotimes_{i=0}^{n-1}\left(\frac{1}{\sqrt{2}}\sum_{k_i,j_i=0}^{1}(-1)^{j_i k_i}|k_i\rangle\langle j_i|\right) \\
&= \frac{1}{2^{n/2}}\sum_{k_0,j_0=0}^{1}\sum_{k_1,j_1=0}^{1}\cdots\sum_{k_{n-1},j_{n-1}=0}^{1}(-1)^{j_0k_0+j_1k_1+\ldots j_{n-1}k_{n-1}} \\
&\times \; |k_0\rangle\otimes|k_1\rangle\cdots|k_{n-1}\rangle\langle j_0|\langle j_1|\cdots\langle j_{n-1}| \\
&= \frac{1}{2^{n/2}}\sum_{k,j=0}^{2^n-1}(-1)^{j\cdot k}|k\rangle\langle j|
\end{aligned}$$

where in the final step, we have used the binary expansions of j,k ($k = k_0\times 2^0 + k_1\times 2^1 + \ldots + k_{n-1}2^{n-1}$) and the definition of the bitwise inner product given in the problem statement. If this seems mysterious, consider that as the individual sums over k_i, j_i run over all values, the tensor product of kets and bras runs over all 2^N binary representations of those 2^n numbers. Try this with a small N to see how it works.

- (h)

$$
\begin{aligned}
H^{\otimes n}\Pi H^{\otimes n} &= \frac{1}{2^{n/2}}\sum_{j,k=0}^{2^n-1}(-1)^{k\cdot j}|k\rangle\langle j|\left(2|0\rangle\langle 0|-I\right)\frac{1}{2^{n/2}}\sum_{l,m=0}^{2^n-1}(-1)^{l\cdot m}|l\rangle\langle m| \\
&= \frac{1}{2^n}\sum_{j,k,l,m=0}^{2^n-1}(-1)^{k\cdot j}(-1)^{l\cdot m}\left(2|k\rangle\langle j|0\rangle\langle 0|l\rangle\langle m|-|k\rangle\langle j|l\rangle\langle m|\right) \\
&= \frac{1}{2^n}\sum_{j,k,l,m=0}^{2^n-1}(-1)^{k\cdot j}(-1)^{l\cdot m}\left(2\delta_{j,0}\delta_{l,0}|k\rangle\langle m|-\delta_{j,l}|k\rangle\langle m|\right) \\
&= \frac{1}{2^n}\sum_{k,m=0}^{2^n-1}(-1)^{k\cdot 0}(-1)^{0\cdot m}2|k\rangle\langle m|-\frac{1}{2^n}\sum_{k,j,m=0}^{2^n-1}(-1)^{(k\oplus m)\cdot j}|k\rangle\langle m| \\
&= 2|s\rangle\langle s|-\frac{1}{2^n}\sum_{k,m=0}^{2^n-1}\delta_{k,m}|k\rangle\langle m| \\
&= 2|s\rangle\langle s|-\frac{1}{2^n}2^n\sum_{k=0}^{2^n-1}|k\rangle\langle k| \\
&= 2|s\rangle\langle s|-I
\end{aligned}
$$

where $k\oplus m$ is performed bitwise. Note that $a\oplus a=0$, so that $(-1)^{(a\oplus a)\cdot j}=1$. Conversely, if we fix k,m and vary j so that $c:=k\oplus m$, then $\sum_j(-1)^{c\cdot j}=0$ as alternating j bitwise will result in an equal number of negative and positive terms. Therefore, the sum over j gives the delta function as written.

- (i)

$$
\begin{aligned}
H^{\otimes n} X_a H^{\otimes n} &= \frac{1}{2^{n/2}} \sum_{j,k=0}^{2^n-1} (-1)^{k\cdot j} |k\rangle\langle j| \left(\sum_{q=0}^{2^n-1} |q \oplus a\rangle\langle q| \right) \frac{1}{2^{n/2}} \sum_{l,m=0}^{2^n-1} (-1)^{l\cdot m} |l\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{j,k,q,l,m=0}^{2^n-1} (-1)^{k\cdot j} (-1)^{l\cdot m} |k\rangle\langle j|q \oplus a\rangle\langle q|l\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{j,k,q,l,m=0}^{2^n-1} (-1)^{k\cdot j} (-1)^{l\cdot m} |k\rangle \delta_{j,q\oplus a} \delta_{q,l} \langle m| \\
&= \frac{1}{2^n} \sum_{k,q,m=0}^{2^n-1} (-1)^{k\cdot (q\oplus a)} (-1)^{q\cdot m} |k\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{k,q,m=0}^{2^n-1} (-1)^{k\cdot a} (-1)^{k\cdot q} (-1)^{q\cdot m} |k\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{k,q,m=0}^{2^n-1} (-1)^{k\cdot a} (-1)^{(k\oplus m)\cdot q} |k\rangle\langle m| \\
&= \frac{1}{2^n} \sum_{k,m=0}^{2^n-1} (-1)^{k\cdot a} \delta_{k,m} |k\rangle\langle m| \\
&= \frac{2^n}{2^n} \sum_{k=0}^{2^n-1} (-1)^{k\cdot a} |k\rangle\langle k| \\
&= Z_a
\end{aligned}
$$

We have utilized properties derived in part (g) to collapse $(-1)^{(k\oplus m)\cdot q}$.

2.3 The Price is Right: *Royal Match*

- (a) Let $P(M_i)$ represent the probability of contestant i getting a "Match" card and $P(NM_i)$ the probability of getting a "No Match card". We assume that the the assignment of cards is independent between contestants and equiprobable, so that $P(M_1) = P(M_2) = P(NM_1) = P(NM_2) = \frac{1}{2}$. There are 4 possible outcomes, but only one is the "Match-Match" case. That means 75% of the time, at least one contestant holds a "No Match" card. If the two contestants agree to always reveal the same card (say King), then they will win 75% of the time.

- (b) Alice and Bob share one of Jason Bourne's entangled qubits: $\psi = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Before analyzing their probability of winning, we first recall that for spin 1 particles, we have

$$
U(\theta) = e^{-i\theta Y} = I\cos\theta - iY\sin\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}
$$

We also consider the action of iY on the basis kets, noting that $iY|0\rangle = -|1\rangle$ and $iY|1\rangle = |0\rangle$.

Let $P(W|\{(M_1, M_2), (M_1, NM_2), (NM_1, M_2), (NM_1, NM_2)\})$ represent the conditional probability of winning given that the players receive the given cards.

1. $P(W|NM_1, NM_2) = 1$

   Alice and Bob both receive "No Match", so a win will occur only if they both reveal King or both reveal Queen. Alice and Bob have the same strategy when they receive a "No Match" card—measure and reveal a King if they see a 0 and a Queen if they see a 1. The probability they both measure 0 is given by the Born rule:
   $$|\langle 00|\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)|^2 = \frac{1}{2}$$
   A similar calculation shows that the probability the both measure 1 is also $\frac{1}{2}$ and there is 0 probability to measure 10 or 01. Thus Alice and Bob always measure the same outcome, always reveal the same cards and will consequently always win if they both receive "No Match" cards.

2. $P(W|NM_1, M_2) = \cos^2(\frac{\pi}{8})$

   Alice receives a "No Match" card and Bob receives a "Match" card. According to their strategies, Bob will rotate his qubit by $\frac{\pi}{8}$ prior to measuring:
   $$\begin{aligned}\psi' &= I \otimes U\left(\frac{\pi}{8}\right)\psi = (I \otimes I \cos\frac{\pi}{8} - iI \otimes Y \sin\frac{\pi}{8})\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}|00\rangle + \cos\frac{\pi}{8}|11\rangle + \sin\frac{\pi}{8}|01\rangle - \sin\frac{\pi}{8}|10\rangle\right)\end{aligned}$$
   After this rotation, Alice and Bob again have the same strategy–revealing King if they measure 0 and Queen if they measure 1. Since Alice still has a "No Match" card, a win occurs only if they reveal the same card. The probability they both measure 0 is given by the Born rule and is just the modulus-squared of $|00\rangle$, which is $\cos^2(\frac{\pi}{8})/2$. A similar calculation gives the same probability of getting $|11\rangle$ so that the probability of winning in this case is $\cos^2(\frac{\pi}{8})$.

3. $P(W|M_1, NM_2) = \cos^2(\frac{\pi}{8})$

   This case is very similar to the previous one. Now Alice receives a "Match" card and Bob receives a "No Match" card. According to their strategies, Alice will rotate her qubit by $-\frac{\pi}{8}$ prior to measuring (recall $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$):
   $$\begin{aligned}\psi' &= U\left(-\frac{\pi}{8}\right) \otimes I\psi = (I \otimes I \cos\frac{\pi}{8} + iY \otimes I \sin\frac{\pi}{8})\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}|00\rangle + \cos\frac{\pi}{8}|11\rangle + \sin\frac{\pi}{8}|01\rangle - \sin\frac{\pi}{8}|10\rangle\right)\end{aligned}$$

After this rotation, Alice and Bob again have the same strategy–revealing King if they measure 0 and Queen if they measure 1. Since Bob still has a "No Match" card, a win occurs only if they reveal the same card. We see that ψ' is the same as the previous case, so the probability of winning is also the same—$\cos^2(\frac{\pi}{8})$.

4. $P(W|M_1, M_2) = \frac{1}{2}$

   Alice and Bob both receive the "Match" card. According to their strategies, both perform the rotations on their qubit, so that

$$\begin{aligned}
\psi' &= \left(U\left(-\frac{\pi}{8}\right) \otimes I\right)\left(I \otimes U\left(\frac{\pi}{8}\right)\right)\psi = U\left(-\frac{\pi}{8}\right) \otimes U\left(\frac{\pi}{8}\right)\psi \\
&= (I\cos\frac{\pi}{8} + iY\sin\frac{\pi}{8}) \otimes (I\cos\frac{\pi}{8} - iY\sin\frac{\pi}{8})\psi \\
&= \left[I \otimes I\cos^2\frac{\pi}{8} + \frac{i}{2}Y \otimes I\sin\frac{\pi}{4} - \frac{i}{2}I \otimes Y\sin\frac{\pi}{4} + Y \otimes Y\sin^2\frac{\pi}{8}\right]\psi \\
&= \frac{1}{\sqrt{2}}\left[\cos^2\frac{\pi}{8}(|00\rangle + |11\rangle) + \frac{\sin\frac{\pi}{4}}{2}(|01\rangle - |10\rangle) + \frac{\sin\frac{\pi}{4}}{2}(|01\rangle - |10\rangle) - \sin^2\frac{\pi}{8}(|00\rangle + |11\rangle)\right] \\
&= \frac{1}{\sqrt{2}}\left[\cos\frac{\pi}{4}|00\rangle + \cos\frac{\pi}{4}|11\rangle + \sin\frac{\pi}{4}|01\rangle - \sin\frac{\pi}{4}|10\rangle\right] \\
&= \frac{1}{2}\left[|00\rangle + |01\rangle - |10\rangle + |11\rangle\right]
\end{aligned}$$

   Since Alice and Bob reveal the same card type if they measure the same thing, they will only reveal a "Royal Match" if they measure different things—$|01\rangle$ or $|10\rangle$. Each outcome in this case is equally likely, so there is a probability of 1/2 of winning in this case.

- (c) The probability of winning, given that the assignment of each card is equiprobable is:

$$\begin{aligned}
P(W) &= P(W|NM_1, NM_2)P(NM_1, NM_2) + P(W|NM_1, M_2)P(NM_1, M_2) \\
&+ P(W|M_1, NM_2)P(M_1, NM_2) + P(W|M_1, M_2)P(M_1, M_2) \\
&= 1 \times \frac{1}{4} + \cos^2\frac{\pi}{8} \times \frac{1}{4} + \cos^2\frac{\pi}{8} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{4} \\
&\approx 80.18\%
\end{aligned}$$

2.4 Extra Credit: Royal Match, Take II

Given the similarity with the previous problem, the following solution will be less detailed.

- (a)

1. $P(W|NM_1, NM_2) = \cos^2\frac{\pi}{8}$

Prior to measuring, Bob will rotate his qubit by $\frac{\pi}{8}$:

$$\begin{aligned}\psi' &= I \otimes U\left(\frac{\pi}{8}\right)\psi = (I \otimes I \cos\frac{\pi}{8} - iI \otimes Y \sin\frac{\pi}{8})\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}|00\rangle + \cos\frac{\pi}{8}|11\rangle + \sin\frac{\pi}{8}|01\rangle - \sin\frac{\pi}{8}|10\rangle\right)\end{aligned}$$

Since Alice and Bob reveal the *same* cards when they measure a 0 or 1, they will now win if they measure bits with the *same* parity. Looking at the coefficients of $|00\rangle$ and $|11\rangle$, we see that the probability of success is $\cos^2\frac{\pi}{8}$.

2. $P(W|NM_1, M_2) = \cos^2\frac{\pi}{8}$

   Prior to measuring, Bob will rotate his qubit by $-\frac{\pi}{8}$:

$$\begin{aligned}\psi' &= I \otimes U\left(-\frac{\pi}{8}\right)\psi = (I \otimes I \cos\frac{\pi}{8} + iI \otimes Y \sin\frac{\pi}{8})\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}|00\rangle + \cos\frac{\pi}{8}|11\rangle - \sin\frac{\pi}{8}|01\rangle + \sin\frac{\pi}{8}|10\rangle\right)\end{aligned}$$

   Since Alice and Bob reveal the *same* cards when they measure a 0 or 1, they will now win if they measure bits with the *same* parity. Looking at the coefficients of $|00\rangle$ and $|11\rangle$, we see that the probability of success is $\cos^2\frac{\pi}{8}$.

3. $P(W|M_1, NM_2) = \cos^2\frac{\pi}{8}$

   Prior to measuring, Alice rotates her qubit by $\frac{\pi}{4}$ and Bob will rotate his qubit by $\frac{\pi}{8}$:

$$\begin{aligned}\psi' &= U\left(\frac{\pi}{4}\right) \otimes U\left(\frac{\pi}{8}\right)\psi = \left(I\cos\frac{\pi}{4} - iY\sin\frac{\pi}{4}\right) \otimes \left(I\cos\frac{\pi}{8} - iY\sin\frac{\pi}{8}\right)\psi \\ &= \left(I \otimes I \cos\frac{\pi}{4}\cos\frac{\pi}{8} - iY \otimes I \sin\frac{\pi}{4}\cos\frac{\pi}{8} - iI \otimes Y \cos\frac{\pi}{4}\sin\frac{\pi}{8} - Y \otimes Y \sin\frac{\pi}{4}\sin\frac{\pi}{8}\right) \\ &\times \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \\ &= \frac{1}{\sqrt{2}}[\cos\frac{\pi}{4}\cos\frac{\pi}{8}(|00\rangle + |11\rangle) + \sin\frac{\pi}{4}\cos\frac{\pi}{8}(|10\rangle - |01\rangle) \\ &\quad + \cos\frac{\pi}{4}\sin\frac{\pi}{8}(|01\rangle - |10\rangle) - \sin\frac{\pi}{4}\sin\frac{\pi}{8}(|00\rangle + |11\rangle)] \\ &= \frac{1}{\sqrt{2}}\left(\cos\frac{\pi}{8}|00\rangle + \cos\frac{\pi}{8}|11\rangle - \sin\frac{\pi}{8}|01\rangle + \sin\frac{\pi}{8}|10\rangle\right)\end{aligned}$$

   Since Alice and Bob reveal the *same* cards when they measure a 0 or 1, they will now win if they measure bits with the *same* parity. Looking at the coefficients of $|00\rangle$ and $|11\rangle$, we see that the probability of success is $\cos^2\frac{\pi}{8}$.

4. $P(W|M_1, M_2) = \cos^2\frac{\pi}{8}$

Prior to measuring, Alice rotates her qubit by $\frac{\pi}{4}$ and Bob will rotate his qubit by $-\frac{\pi}{8}$:

$$
\begin{aligned}
\psi' &= U\left(\frac{\pi}{4}\right) \otimes U\left(-\frac{\pi}{8}\right) \psi = \left(I \cos\frac{\pi}{4} - iY \sin\frac{\pi}{4}\right) \otimes \left(I \cos\frac{\pi}{8} + iY \sin\frac{\pi}{8}\right) \psi \\
&= \left(I \otimes I \cos\frac{\pi}{4} \cos\frac{\pi}{8} - iY \otimes I \sin\frac{\pi}{4} \cos\frac{\pi}{8} + iI \otimes Y \cos\frac{\pi}{4} \sin\frac{\pi}{8} + Y \otimes Y \sin\frac{\pi}{4} \sin\frac{\pi}{8}\right) \\
&\times \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \\
&= \frac{1}{\sqrt{2}} [\cos\frac{\pi}{4} \cos\frac{\pi}{8} (|00\rangle + |11\rangle) + \sin\frac{\pi}{4} \cos\frac{\pi}{8} (|10\rangle - |01\rangle) \\
&\quad + \cos\frac{\pi}{4} \sin\frac{\pi}{8} (|10\rangle - |01\rangle) - \sin\frac{\pi}{4} \sin\frac{\pi}{8} (|00\rangle + |11\rangle)] \\
&= \frac{1}{\sqrt{2}} \left(\cos\frac{3\pi}{8}|00\rangle + \cos\frac{3\pi}{8}|11\rangle - \sin\frac{3\pi}{8}|01\rangle + \sin\frac{3\pi}{8}|10\rangle\right)
\end{aligned}
$$

Since Alice and Bob reveal the *same* cards when they measure a 0 or 1, they will now win if they measure bits with *different* parity (to get the royal match). Looking at the coefficients of $|01\rangle$ and $|10\rangle$, we see that the probability of success is $\sin^2 \frac{3\pi}{8} = \cos^2 \frac{\pi}{8}$.

- (c) The probability of winning, given that the assignment of each card is equiprobable is:

$$
\begin{aligned}
P(W) &= P(W|NM_1, NM_2)P(NM_1, NM_2) + P(W|NM_1, M_2)P(NM_1, M_2) \\
&+ P(W|M_1, NM_2)P(M_1, NM_2) + P(W|M_1, M_2)P(M_1, M_2) \\
&= \frac{1}{4}(4\cos^2\frac{\pi}{8}) \\
&\approx 85.36\%
\end{aligned}
$$