UNM Physics 452/581: Introduction to Quantum Information, Solution Set 5, Fall 2007

5.1 Exact Grover for 1 in N Ordered Search

- (a) For $N = 1$ and 4, $\bar{k}$ is an integer.

N	$\bar{k} = \frac{\pi}{4\alpha} - \frac{1}{2}$
1	0
2	$\frac{1}{2}$
3	$-\frac{1}{2} + \frac{\pi}{4\mathrm{ArcSin}\left[\frac{1}{\sqrt{3}}\right]} \approx 0.776$
4	1

- (b) In order to B to be unitary, $B^\dagger B = I$. We will assume N and a are real and non-negative.

$$\begin{aligned}
1 &= \langle 0|0\rangle && (1)\\
&= \langle 0|B^\dagger B|0\rangle && (2)\\
&= \left(\sqrt{1-Na}\langle 0| + \sqrt{Na}\langle 1|\right)\left(\sqrt{1-Na}|0\rangle + \sqrt{Na}|1\rangle\right) && (3)\\
&= \|\sqrt{(1-Na)}\|^2\langle 0|0\rangle + \|Na\|^2\langle 1|1\rangle && (4)\\
&= 1 && (5)
\end{aligned}$$

For this last equality to hold, Na must be between 0 and 1. This can also be seen by noting that the amplitudes must give probabilities that are between 0 and 1. For instance, $\|\langle 0|B|0\rangle\|^2 = Na$ and $\|\langle 1|B|0\rangle\|^2 = 1 - Na$ must be between 0 and 1. Thus, a should be between 0 and $\frac{1}{N}$. If a were possibly negative, then $-\frac{1}{N} \le a \le \frac{1}{N}$.

- (c) Symmetry suggests

$$B|1\rangle = -\sqrt{Na}|0\rangle + \sqrt{1-Na}|1\rangle \tag{6}$$

We then check

$$\begin{aligned}
1 &= \langle 1|1\rangle && (7)\\
&= \left(-\sqrt{Na}\langle 0| + \sqrt{1-Na}\langle 1|\right)\left(-\sqrt{Na}|0\rangle + \sqrt{1-Na}|1\rangle\right) && (8)\\
&= Na\langle 0|0\rangle + (1-Na)\langle 1|1\rangle && (9)\\
&= 1 && (10)
\end{aligned}$$

and

$$\begin{aligned}
0 &= \langle 0|1\rangle && (11)\\
&= \left(-\sqrt{Na}\langle 0| + \sqrt{1-Na}\langle 1|\right)\left(\sqrt{1-Na}|0\rangle + \sqrt{Na}|1\rangle\right) && (12)\\
&= -\sqrt{Na}\sqrt{1-Na}\langle 0|0\rangle + \sqrt{Na}\sqrt{1-Na}\langle 1|1\rangle && (13)\\
&= 0 && (14)
\end{aligned}$$

That $\langle 1|0\rangle = 0$ follows from this last result.

- (d) Calculating explicitly (and using $W|0\rangle^{\otimes n} = \sqrt{1-1/N}|w^\perp\rangle + \sqrt{1/N}|w\rangle$ from earlier in the problem)

$$\begin{aligned}
W'|0\rangle^{\otimes(n+1)} &= B \otimes H^{\otimes n}|0\rangle \otimes |0\rangle^{\otimes n} && (15)\\
&= \left[\sqrt{1-Na}|0\rangle + \sqrt{Na}|1\rangle\right] \otimes \left[\sqrt{1-\frac{1}{N}}|w^\perp\rangle + \sqrt{\frac{1}{N}}|w\rangle\right] && (16)\\
&= \underbrace{\sqrt{(1-Na)(1-\frac{1}{N})}|0\rangle|w^\perp\rangle + \sqrt{Na-a}|1\rangle|w^\perp\rangle + \sqrt{\frac{1}{N}-a}|0\rangle|w\rangle}_{:=\cos\alpha'|w'^\perp\rangle} && (17)\\
&+ \underbrace{\sqrt{a}|1\rangle|w\rangle}_{=\sin\alpha'|w'\rangle} && (18)\\
&= \cos\alpha'|w'^\perp\rangle + \sin\alpha'|w'\rangle && (19)\\
&= |s'\rangle && (20)
\end{aligned}$$

where we have simply lumped the terms orthogonal to $|w'\rangle = |0\rangle|w\rangle$ into an orthogonal term $|w'^\perp\rangle$ with the appropriate phase.

- (e) We write

$$Z_X' = I - 2|w'\rangle\langle w'| = I - 2|1\rangle\langle 1| \otimes |w\rangle\langle w| \tag{21}$$

If the first qubit is in the state $|0\rangle$, then $Z_X' = I$. If it is in state $|1\rangle$, then Z_X' performs Z_X on the remaining qubits. By definition, this is the controlled gate, $\Lambda(Z_X)$.

- (f) Using the definitions given in the problem, we calculate

$$\begin{aligned}
a = \sin^2\alpha' &= \sin^2\frac{\pi}{4\lceil\bar{k}\rceil+2} && (22)\\
&\le \sin^2\frac{\pi}{4\lceil\bar{k}\rceil+2} && (23)\\
&= \sin^2\alpha && (24)\\
&= \frac{1}{N} && (25)
\end{aligned}$$

- (g) In order to yield $|w'\rangle$ with certainty, we need $\sin((2k+1)\alpha') = 1$ or $(2k+1)\alpha' = \pi/2$. Setting $k = \lceil\bar{k}\rceil$, we find

$$(2\lceil\bar{k}\rceil+1)\alpha' = \frac{(2\lceil\bar{k}\rceil+1)\pi}{(4\lceil\bar{k}\rceil+2)} = \frac{(2\lceil\bar{k}\rceil+1)\pi}{2(2\lceil\bar{k}\rceil+1)} = \frac{\pi}{2} \tag{26}$$

5.2 The phase estimation algorithm

- (a) At the beginning of round j, the input state is $|0\rangle|u\rangle$. Passing through the first Hadamard gives

$$\frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right]|u\rangle \tag{27}$$

The controlled $U^{2^{k_j}}$ gate gives the state

$$\frac{1}{\sqrt{2}}\left[|0\rangle + \exp(2\pi i\phi 2^{k_j})|1\rangle\right]|u\rangle \tag{28}$$

Since the $|u\rangle$ can be factored out for the rest of the circuit, we omit it for the rest of the calculation. Noting that $\exp(-i\theta Z/2)|0\rangle = \exp(-i\theta/2)|0\rangle$ and $\exp(-i\theta Z/2)|1\rangle = \exp(i\theta/2)|1\rangle$, we find

$$\begin{aligned}
& e^{-i\theta_j Z/2}\frac{1}{\sqrt{2}}\left[|0\rangle + \exp(2\pi i\phi 2^{k_j})|1\rangle\right] && (29)\\
&= \frac{1}{\sqrt{2}}\left[e^{-i\theta_j/2}|0\rangle + \exp(2\pi i\phi 2^{k_j} + i\theta_j/2)|1\rangle\right] && (30)\\
&\equiv \frac{1}{\sqrt{2}}\left[|0\rangle + \exp(2\pi i\phi 2^{k_j} + i\theta_j)|1\rangle\right] && (31)\\
& && (32)
\end{aligned}$$

where in the last step we have factored out an irrelevant overall phase $e^{-i\theta_j/2}$. The second Hadamard transforms the state to

$$\begin{aligned}
& \frac{1}{\sqrt{2}}\left[H|0\rangle + \exp(2\pi i\phi 2^{k_j} + i\theta_j)H|1\rangle\right] && (33)\\
&= \frac{1}{2}\left[|0\rangle + |1\rangle + \exp(2\pi i\phi 2^{k_j} + i\theta_j)(|0\rangle - |1\rangle\right] && (34)\\
&= \frac{1}{2}\left[\left(1 + \exp(2\pi i\phi 2^{k_j} + i\theta_j)\right)|0\rangle + \left(1 - \exp(2\pi i\phi 2^{k_j} + i\theta_j)\right)|1\rangle\right] && (35)
\end{aligned}$$

The probability of measuring z_j is given by the norm of the coefficient of each $|j\rangle$ term above. Thus the probabilities are

$$\begin{aligned}
P(z_j = 0) &= \frac{1}{4}\|1 + \exp\left[i(2\pi\phi 2^{k_j} + \theta_j)\right]\|^2 && (36)\\
P(z_j = 1) &= \frac{1}{4}\|1 - \exp\left[i(2\pi\phi 2^{k_j} + \theta_j)\right]\|^2 && (37)
\end{aligned}$$

- (b) Recall that

$$\phi = 0.\phi_1\phi_2\ldots\phi_n = \frac{1}{2}\phi_1 + \frac{1}{2^2}\phi_2 + \cdots + \frac{1}{2^n}\phi_n \tag{38}$$

Taking $k_1 = n-1$, we have

$$2^{n-1}\phi = \phi_1\phi_2\ldots\phi_{n-1}.\phi_n \tag{39}$$

Expanding the exponent into products, we have

$$\underbrace{\left(\prod_{m=1}^{n-1} e^{i2^{n-m}\pi\phi_m}\right)}_{=1} e^{i\pi\phi_n + i\theta_1} \tag{40}$$

Setting $\theta_1 = 0$, the probabilities are then

$$P(z_1 = 0) = \frac{1}{4}\|1 + \exp[i\pi\phi_n]\|^2 \tag{41}$$

$$P(z_1 = 1) = \frac{1}{4}\|1 - \exp[i\pi\phi_n]\|^2 \tag{42}$$

So that if $\phi_n = 0$, $P(z_1 = 0)$ is one and $P(z_1 = 1)$ is zero. If $\phi_n = 1$, $P(z_1 = 0)$ is zero and $P(z_1 = 1)$ is one. Thus with probability one, $z_1 = \phi_n$.

- (c) Inspired by part (b), we proceed in a similar manner by choosing $k_2 = n - 2$ so that

$$2^{n-2}\phi = \phi_1\phi_2\ldots\phi_{n-2}.\phi_{n-1}\phi_n \tag{43}$$

Expanding the exponent into products again , we have

$$\underbrace{\left(\prod_{m=1}^{n-2} e^{i2^{n-m-1}\pi\phi_m}\right)}_{=1} e^{i\pi\phi_{n-1}+i\frac{\pi}{2}\phi_n+i\theta_1} \tag{44}$$

Since we *know* ϕ_n with certainty, we choose $\theta_1 = -\frac{\pi}{2}\phi_n$, giving probabilities

$$P(z_2 = 0) = \frac{1}{4}\|1 + \exp[i\pi\phi_{n-1}]\|^2 \tag{45}$$

$$P(z_2 = 1) = \frac{1}{4}\|1 - \exp[i\pi\phi_{n-1}]\|^2 \tag{46}$$

As in part (b), these yield $z_2 = \phi_{n-1}$ with certainty.

- (d) Given the last two steps, the general procedure is as follows. On step j, choose $k_j = n - j$ so that

$$2^{n-j}\phi = \phi_1\phi_2\ldots\phi_{n-j}.\phi_{n-j+1}\ldots\phi_n \tag{47}$$

Expanding the exponent into products again, we have

$$\underbrace{\left(\prod_{m=1}^{n-j} e^{i2^{n-m-(j-1)}\pi\phi_m}\right)}_{=1} \exp\left[i\pi\phi_{n-j+1} + i\frac{\pi}{2}\phi_{n-j+2} + \ldots + i\frac{\pi}{2^{1-j}}\phi_n + i\theta_j\right] \tag{48}$$

Since we know ϕ_{n-j+2} through ϕ_n at stage j, we choose

$$\theta_j = -\pi\left[\frac{\phi_{n-j+2}}{2} + \frac{\phi_{n-j+3}}{2^2} + \ldots + \frac{\phi_n}{2^{1-j}}\right] \tag{49}$$

to cancel out all but the ϕ_{n-j+1} exponential. This gives probabilities

$$P(z_j = 0) = \frac{1}{4}\|1 + \exp[i\pi\phi_{n-j+1}]\|^2 \tag{50}$$

$$P(z_j = 1) = \frac{1}{4}\|1 - \exp[i\pi\phi_{n-j+1}]\|^2 \tag{51}$$

so that $z_j = \phi_{n-j+1}$ with certatinty.