UNM Physics 452/581: Introduction to Quantum Information, Problem Set 6, Fall 2007

Instructor: Dr. Landahl
Issued: Monday, November 12, 2007
Due: Tuesday, November 27, 2007

Do all of the problems listed below. Hand in your problem set at the beginning of class on the desk at the front of the classroom or after class in the box in the Physics and Astronomy main office by 5 p.m. **Please put your name and/or IQI number number on your assignment**, as well as the course number (Physics 452/581). Please show all your work and write clearly. Credit will be awarded for clear explanations as much, if not more so, than numerical answers. Avoid the temptation to simply write down an equation and move symbols around or plug in numbers. Explain what you are doing, draw pictures, and check your results using common sense, limits, and/or dimensional analysis.

6.1. Quantum division.

The subsystems of a bipartite system can have different dimensions. Consider the following tripartite states:

$$|GHZ\rangle := \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle), \qquad |W\rangle := \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle).$$

Let the names of the three subsystems be A, B, and C. (By symmetry, it doesn't matter which is which.)

(a) Compute ρ_{AB} and ρ_C for $|GHZ\rangle$ and for $|W\rangle$.

(b) Compute the Schmidt decomposition over the partition $AB|C$ for $|GHZ\rangle$ and for $|W\rangle$.

(c) Compute the eigenvalues of ρ_{AB} and ρ_C for $|GHZ\rangle$ and for $|W\rangle$. Notice anything interesting about how the eigenvalues of ρ_{AB} and ρ_C are related?

6.2. Quantum tintinnabulation.

The *Bell states* are a basis over two qubits consisting of the four entangled states

$$|\Phi^\pm\rangle := \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \qquad |\Psi^\pm\rangle := \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle).$$

A *Werner state* of fidelity F is the mixed state

$$\rho_F := F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}(|\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|).$$

(a) Show that the depolarizing channel with error probability p acting on half of the entangled state $|\Phi^+\rangle$ yields a Werner state. Express the fidelity of the Werner state F in terms of p.

(b) Show that the Werner state from part (a) can also be written as

$$\rho_F := \lambda|\Phi^+\rangle\langle\Phi^+| + \frac{1-\lambda}{4}(I \otimes I),$$

and express λ in terms of F.

6.3. Holy Schmidt!

Find the Schmidt decomposition and calculate ρ_A for each of the following states. (Note: The last two parts will probably take more work than the first three.)

(a) $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$.

(b) $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$.

(c) $\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$.

(d) $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$.

(e) $CNOT(H \otimes Y_{-2\pi/3})|00\rangle$, where $Y_{-2\pi/3}$ is the spin-1/2 rotation matrix about the $\hat{y}$-axis by the angle $-2\pi/3$.

6.4. Trace distance.

In class we defined the *trace distance* between two density matrices as

$$D(\rho, \sigma) := \frac{1}{2}\|\rho - \sigma\|_{\mathrm{tr}},$$

where the trace norm is defined by

$$\|A\|_{\mathrm{tr}} := \mathrm{tr}\,\sqrt{A^\dagger A}. \tag{1}$$

(a) Show that the trace distance between density matrices ρ and σ is equal to half the sum of the absolute values of the eigenvalues of $\rho - \sigma$.

(b) Compute the trace distance between $\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|$ and $\frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|$.

(c) Compute the trace distance between two arbitrary pure states as a function of the angle θ between them. Note that without loss of generality these two states can be expressed as $|\psi\rangle = \sin\theta/2|e_0\rangle + \cos\theta/2|e_1\rangle$ and $|\varphi\rangle = -\sin\theta/2|e_0\rangle + \cos\theta/2|e_1\rangle$ by a suitable choice of basis $\{|e_0\rangle, |e_1\rangle\}$.

(d) Compute $\||\psi\rangle - |\varphi\rangle\|^2$ in terms of θ, where $\|\cdot\|$ denotes the usual complex vector norm on quantum states. Use this result and the result of part (b) to derive the following bound on pure state distances:

$$D(|\psi\rangle\langle\psi|, |\varphi\rangle\langle\varphi|) \le \||\psi\rangle - |\varphi\rangle\|.$$

(e) Given the bound from part (c), it is tempting to use $\||\psi\rangle - |\varphi\rangle\|$ as a measure of distance between pure states. Explain why this is probably not a good idea. (*Hint*: When are two pure states indistinguishable?)

6.5. Double damplitude.

In class, we defined the amplitude-damping channel $\mathcal{A}$ as the channel with Kraus operators

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix},$$

where $p = 1 - e^{-t/T_1}$.

Let $f(|\psi\rangle)$ denote the fidelity $F(|\psi\rangle\langle\psi|,\ \mathcal{A}(|\psi\rangle\langle\psi|))$ between a pure state $|\psi\rangle$ and its output under the amplitude damping channel.

(a) Compute $f(|+\rangle)$ as a function of p. Plot the derived function $f(t)$. What is $\lim_{t\to\infty} f(t)$?

(b) Use calculus to find the qubit state $|\psi\rangle$ that minimizes $f(|\psi\rangle)$. Compute $f(|\psi\rangle)$. Plot the derived function $f(t)$. What is $\lim_{t\to\infty} f(t)$?

A *dual-rail* encoding of a qubit is the map

$$|0\rangle \mapsto |\overline{0}\rangle := |01\rangle, \qquad |1\rangle \mapsto |\overline{1}\rangle := |10\rangle.$$

It allows a qubit to be encoded, for example, by a photon that is in one of two possible modes. In the following, let $|\varphi\rangle$ denote the dual-rail qubit state $\alpha|\overline{0}\rangle + \beta|\overline{1}\rangle$, where α and β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$. Also let $g(|\varphi\rangle)$ denote the fidelity $F(|\varphi\rangle\langle\varphi|,\ (\mathcal{A}\otimes\mathcal{A})(|\varphi\rangle\langle\varphi|))$.

(c) Show that $\mathcal{A}\otimes\mathcal{A}$ acting on $|\varphi\rangle$ has the same effect as the quantum operation whose Kraus operators are

$$E_0 = \sqrt{1-p}\, I, \qquad E_1 = \sqrt{p}\, |00\rangle\langle 01|, \qquad E_2 = \sqrt{p}\, |00\rangle\langle 10|.$$

(Note: These Kraus operators describe a trace-decreasing map, so they don't obey the normalization condition $\sum_i E_i^\dagger E_i = I$ discussed in class—don't worry about this for this problem.)

(d) Compute $g(|\varphi\rangle)$ as a function of p for the worst-case $|\varphi\rangle$. How does this worst-case g compare to the worst-case f from part (b)?

(e) Suppose one measures the observable ZZ after $\mathcal{A}\otimes\mathcal{A}$ has acted on $|\varphi\rangle$. (For a photonic implementation, this is akin to measuring the total photon number.) What is the probability of obtaining the outcome $|00\rangle$? When this outcome is not obtained, what is the resulting state, and what is its fidelity with respect to $|\varphi\rangle$ as a function of t? Considering these results, explain why this dual-rail encoding is called an error-detecting code for the amplitude-damping channel.

6.6. Extra credit: High fidelity.

(a) (Warmup I.) Show that the length of a qubit's Bloch vector $\vec{r}$ is related to the density matrix ρ describing it as:

$$|\vec{r}| = \sqrt{1 - 4\det\rho}.$$

(*b*) (Warmup II.) Show that the characteristic equation for the eigenvalues of any 2×2 matrix M can be written as

$$\lambda^2 - \lambda \operatorname{tr} M + \det M = 0.$$

Use this equation to express the trace and determinant of a 2×2 matrix M in terms of its eigenvalues λ_1 and λ_2.

(*c*) Show that the (squared) fidelity between two qubit states can be expressed as

$$F^2(\rho, \sigma) = \frac{1}{2}\left(1 + \vec{r} \cdot \vec{s} + \sqrt{(1 - r^2)(1 - s^2)}\right),$$

where $\vec{r}$ and $\vec{s}$ are the Bloch vectors for the density matrices ρ and σ respectively. (*Hint*: Use what you learned in the "Warmup" parts to this problem.)

(*d*) In class, it was claimed that the fidelity $F(\rho, \sigma) := \operatorname{tr} \sqrt{\rho^{1/2} \sigma \rho^{1/2}}$ is symmetric in its arguments. Written this way, the symmetry is by no means obvious. One way to make it clearer is to rewrite the fidelity in terms of the *trace norm*:

$$F(\rho, \sigma) := \|\rho^{1/2} \sigma^{1/2}\|_{\mathrm{tr}},$$

where

$$\|A\|_{\mathrm{tr}} := \operatorname{tr} \sqrt{A^\dagger A}. \tag{2}$$

Show that the fidelity is symmetric in its arguments by showing that for any Hermitian matrices A and B,

$$\|AB\|_{\mathrm{tr}} = \|BA\|_{\mathrm{tr}},$$

(*Hint*: Show that $ABBA$ and $BAAB$ have the same eigenvalues.)

6.7. Extra credit: Miscellaneous channel problems.

(*a*) Recall from problem 2.2 (*i*) in the second problem set the n-qubit operators X_a and Z_a defined as

$$X_a := \sum_{j=0}^{2^n - 1} |j \oplus a\rangle\langle j|, \qquad Z_a := \sum_{j=0}^{2^n - 1} (-1)^{j \cdot a} |j\rangle\langle j|.$$

Show that the quantum operation having 4^n Kraus operators $X_a Z_b / 2^{n/2}$ acting on a n-qubit density matrix ρ will completely randomize it (*i.e.*, it maps ρ to $I/2^n$). (*Hint*: Consider the action of the channel on $|i\rangle\langle j|$.)

(*b*) Consider the quantum operation having the three Kraus operators

$$A_0 = \frac{\sqrt{5}}{2\sqrt{77}} \begin{pmatrix} 3 & -3\sqrt{2} \\ \sqrt{2} & -2 \end{pmatrix}, \qquad A_1 = \frac{\sqrt{5}}{2\sqrt{77}} \begin{pmatrix} 3 & 3\sqrt{2} \\ -\sqrt{2} & -2 \end{pmatrix}, \qquad A_2 = \frac{1}{\sqrt{14}} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

There are two distinguishable input pure states that are taken to output pure states by this quantum operation. What are these input pure states? (*Hint*: For the output on a pure state $|\psi\rangle$ to be pure, each of the $A_i|\psi\rangle$ must be proportional.)

(c) Let $\mathcal{E}$ be a quantum operation that preserves two distinct non-orthogonal pure states. Prove that it must be the case that $\mathcal{E}$ is the identity map on the space spanned by these states.

(d) Suppose a qubit first transforms under the phase-damping channel then transforms under the amplitude-damping channel. Show that the linear dependence of the operators one obtains for this net channel allows one to represent it by only three Kraus operators. (*Hint*: Start with a compact OSR for the phase-damping channel.)