UNM Physics 452/581: Introduction to Quantum Information, Solution Set 6, Fall 2007

6.1 Quantum division

- (*a*) $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$. In the following, I take A,B,C to represent the first, second and third qubit respectively.

  – First consider $|GHZ\rangle$.

$$\rho_{AB}^{(GHZ)} = \text{tr}_C\, |GHZ\rangle\langle GHZ| = \sum_{i=0}^{1} {}_C\langle i|GHZ\rangle\langle GHZ|i\rangle_C \tag{1}$$
$$= {}_C\langle 0|GHZ\rangle\langle GHZ|0\rangle_C + {}_C\langle 1|GHZ\rangle\langle GHZ|1\rangle_C \tag{2}$$
$$= \frac{1}{2}\, {}_C\langle 0|\Big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big)|0\rangle_C \tag{3}$$
$$+ \frac{1}{2}\, {}_C\langle 1|\Big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big)|1\rangle_C \tag{4}$$
$$= \frac{1}{2}\Big(|00\rangle\langle 00| + |11\rangle\langle 11|\Big) \tag{5}$$

  and

$$\rho_{C}^{(GHZ)} = \text{tr}_C\, |GHZ\rangle\langle GHZ| = \sum_{i=00}^{11} {}_{AB}\langle i|GHZ\rangle\langle GHZ|i\rangle_{AB} \tag{6}$$
$$= {}_{AB}\langle 00|GHZ\rangle\langle GHZ|00\rangle_{AB} + {}_{AB}\langle 01|GHZ\rangle\langle GHZ|01\rangle_{AB} \tag{7}$$
$$+ {}_{AB}\langle 10|GHZ\rangle\langle GHZ|10\rangle_{AB} + {}_{AB}\langle 11|GHZ\rangle\langle GHZ|11\rangle_{AB} \tag{8}$$
$$= \frac{1}{2}{}_{AB}\langle 00|\Big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big)|00\rangle_{AB} \tag{9}$$
$$+ \frac{1}{2}{}_{AB}\langle 11|\Big(|000\rangle\langle 000| + |000\rangle\langle 111| + |111\rangle\langle 000| + |111\rangle\langle 111|\Big)|11\rangle_{AB} \tag{10}$$
$$= \frac{1}{2}\Big(|0\rangle\langle 0| + |1\rangle\langle 1|\Big) \tag{11}$$

  Where in going to (9), I have omitted the $|01\rangle_{AB}$ and $|10\rangle_{AB}$ terms which go to zero.

  – Now consider $|W\rangle$. Since we are starting to get the hang of the partial trace, I will begin to drop more intermediary terms. In particular, I will immediately drop terms from the density matrix that will go to zero.

$$\rho_{AB}^{(W)} = \text{tr}_C |W\rangle\langle W| = \sum_{i=0}^{1} {}_C\langle i|W\rangle\langle W|i\rangle_C \tag{12}$$

$$= \frac{1}{3}\sum_{i=0}^{1} {}_C\langle i|\Big(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100| + |010\rangle\langle 100| + |100\rangle\langle 010|\Big)|i\rangle_C$$

$$= \frac{1}{3}\Big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |01\rangle\langle 10| + |10\rangle\langle 01|\Big) \tag{13}$$

and

$$\rho_C^{(W)} = \text{tr}_{AB} |W\rangle\langle W| = \sum_{i=00}^{11} {}_{AB}\langle i|W\rangle\langle W|i\rangle_{AB} \tag{14}$$

$$= \frac{1}{3}\sum_{i=00}^{11} {}_{AB}\langle i|\Big(|001\rangle\langle 001| + |010\rangle\langle 010| + |100\rangle\langle 100|\Big)|i\rangle_{AB} \tag{15}$$

$$= \frac{1}{3}\Big(|1\rangle\langle 1| + 2|0\rangle\langle 0|\Big) \tag{16}$$

- (b) While we could work out the singular value decomposition explicitly, the structure of these two state make it easy enough to write the Schmidt decomposition directly.
  - $AB|C$ partition for $|GHZ\rangle$.

    $$|GHZ\rangle = \frac{1}{\sqrt{2}}|00\rangle_{AB}|0\rangle_C + \frac{1}{\sqrt{2}}|11\rangle_{AB}|1\rangle_c \tag{17}$$

    So the basis $\mathcal{B}^{(GHZ)}$ for AB is $\mathcal{B}_{AB}^{(GHZ)} = \{|00\rangle, |11\rangle\}$ and for C is $\mathcal{B}_C^{(GHZ)} = \{|0\rangle, |1\rangle\}$ with Schmidt coefficients $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$.
  - $AB|C$ parition for $|W\rangle$.

    $$|W\rangle = \frac{1}{\sqrt{3}}\left(|001\rangle + |010\rangle + |100\rangle\right) \tag{18}$$

    $$= \frac{1}{\sqrt{3}}|00\rangle_{AB}|1\rangle_C + \frac{1}{\sqrt{3}}\left(|01\rangle_{AB} + |10\rangle_{AB}\right)|0\rangle_C \tag{19}$$

    $$= \frac{1}{\sqrt{3}}|00\rangle_{AB}|1\rangle_C + \sqrt{\frac{2}{3}}|\Psi^+\rangle_{AB}|0\rangle_C \tag{20}$$

    $$\tag{21}$$

    where $|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right)$. The Schmidt bases are $\mathcal{B}_{AB}^{(W)} = \{|00\rangle, |\Psi^+\rangle\}$ and $\mathcal{B}_C^{(W)} = \{|1\rangle, |0\rangle\}$ and the Schmidt coefficients are $\{\sqrt{\frac{1}{3}}, \sqrt{\frac{2}{3}}\}$.
- (c)

– For the $|GHZ\rangle$ state, both reduced density matrices are already diagonal in the computational basis. We read off the eigenvalues of $\rho_{AB}^{(GHZ)}$ as $\{\frac{1}{2}, 0, 0, \frac{1}{2}\}$. Similarly, for $\rho_C^{(GHZ)}$, we have $\{\frac{1}{2}, \frac{1}{2}\}$.

– For $|W\rangle$, only $\rho_C^{(W)}$ is diagonal in the computational basis. We can read of the eigenvalues for it as $\{\frac{1}{3}, \frac{2}{3}\}$. $\rho_{AB}^{(W)}$ is a bit trickier, but we can use the Schmidt composition to calculate the diagonal form. If we let $|i\rangle_{AB}$ and $|i\rangle_C$ represent the i-th element of the two different Schmidt bases and λ_i the corresponding (real) Schmidt coefficient, we have

$$\rho_{AB}^{(W)} = \sum_i {}_C\langle i|W\rangle\langle W|i\rangle_C \tag{22}$$

$$= \sum_i {}_C\langle i|\Big(\sum_j \lambda_j^2{}_{AB}|j\rangle_C|j\rangle\langle j|_{AB}\langle j|_C\Big)|i\rangle_C \tag{23}$$

$$= \sum_{i,j} \lambda_j^2{}_{AB}|j\rangle\langle j|_{AB}\|{}_C\langle i|j\rangle_C\|^2 \tag{24}$$

$$= \sum_{i,j} \lambda_j^2{}_{AB}|j\rangle\langle j|_{AB}\delta_{ij} \tag{25}$$

so that, in the basis $\mathcal{B}_{AB}^W$ defined in part (b), we have

$$\rho_{AB}^{(W)} = \frac{1}{3}|00\rangle\langle 00|_{AB} + \frac{2}{3}|\Psi^+\rangle\langle\Psi^+|_{AB} \tag{26}$$

Thus the eigenvalues are $\{\frac{1}{3}, \frac{2}{3}, 0, 0\}$.

From this last calcuation, we notice that both reduced density matrices have the same non-zero eigenvalues, which are just the Schmidt coefficients squared.

6.2 Quantum tintinabulation

- (a) From class, recall that the depolarizing channel acts as

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z\right) \tag{27}$$

Clearly, the $(1-p)$ term will leave the input state unchanged; it remains to see what the Pauli terms do to the input state.

$$X|\Phi^+\rangle\langle\Phi^+|X = \frac{1}{2}X\Big(|00\rangle + |11\rangle\Big)\Big(\langle 00| + \langle 11|\Big)X \tag{28}$$

$$= \frac{1}{2}\Big(|10\rangle + |01\rangle\Big)\Big(\langle 10| + \langle 01|\Big) \tag{29}$$

$$= |\Psi^+\rangle\langle\Psi^+| \tag{30}$$

and

$$Y|\Phi^+\rangle\langle\Phi^+|Y = \frac{1}{2}Y\Big(|00\rangle + |11\rangle\Big)\Big(\langle 00| + \langle 11|\Big)Y \tag{31}$$

$$= \frac{1}{2}\Big(i|10\rangle - i|01\rangle\Big)\Big(-i\langle 10| + i\langle 01|\Big) \tag{32}$$

$$= |\Psi^-\rangle\langle\Psi^-| \tag{33}$$

and

$$Z|\Phi^+\rangle\langle\Phi^+|Z = \frac{1}{2}Z\Big(|00\rangle + |11\rangle\Big)\Big(\langle 00| + \langle 11|\Big)Z \tag{34}$$
$$= \frac{1}{2}\Big(|00\rangle - |11\rangle\Big)\Big(\langle 00| - \langle 11|\Big) \tag{35}$$
$$= |\Phi^-\rangle\langle\Phi^-| \tag{36}$$

Therefore,

$$\mathcal{E}(|\Phi^+\rangle\langle\Phi^+|) = (1-p)|\Phi^+\rangle\langle\Phi^+| + \frac{p}{3}\Big(|\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\Big) \tag{37}$$

so that $F = 1 - p$.

- (b) Note that

$$I \otimes I = |00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|$$
$$= |\Phi^+\rangle\langle\Phi^+| + |\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-| \tag{38}$$

$$\rho_F = F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}\Big(|\Phi^-\rangle\langle\Phi^-| + |\Psi^+\rangle\langle\Psi^+| + |\Psi^-\rangle\langle\Psi^-|\Big) \tag{39}$$
$$= F|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}\Big(I \otimes I - |\Phi^+\rangle\langle\Phi^+|\Big) \tag{40}$$
$$= \frac{4F-1}{3}|\Phi^+\rangle\langle\Phi^+| + \frac{1-F}{3}I \otimes I \tag{41}$$

Reading off, we see that $\lambda = \frac{4F-1}{3}$.

6.3 Holy Schmidt!

In the following, I will use the result from question 5.3c, where we found that the reduced density matrix (ρ_A) for a Schmidt decomposition $\sum_i \lambda_i |i\rangle_A |i\rangle_B$ is simply $\sum_i \lambda_i^2{}_A|i\rangle\langle i|_A$

- (a)

$$\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \frac{1}{\sqrt{2}}|0\rangle_A|0\rangle_B - \frac{1}{\sqrt{2}}|1\rangle_A|1\rangle_B \tag{42}$$

The two basis are $\{|0\rangle_A, |1\rangle_A\}$ and $\{|0\rangle_B, -|1\rangle_B\}$ and the coefficients are $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$. Finally

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{43}$$

- (b)

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{2}(|0\rangle_A(|0\rangle_B + |1\rangle_B) + |1\rangle_A(|0\rangle_B + |1\rangle_B)) \tag{44}$$
$$= \frac{1}{2}\left(|0\rangle_A + |1\rangle_A\right) \otimes \left(|0\rangle_B + |1\rangle_B\right) \tag{45}$$
$$= |+\rangle_A|+\rangle_B \tag{46}$$

Since this is a pure state, we have only one element in the Schmidt decomposition, which is the $|+\rangle$ state for both sub-systems. The corresponding Schmidt coefficient is 1. Finally

$$\rho_A = |+\rangle\langle+| = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|) \tag{47}$$

- (c)

$$\frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle) = \frac{1}{2}(|0\rangle_A(|0\rangle_B + |1\rangle_B) + |1\rangle_A(|0\rangle_B - |1\rangle_B)) \tag{48}$$

$$= \frac{1}{\sqrt{2}}|0\rangle_A|+\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_A|-\rangle_B \tag{49}$$

The basis for system A is then $\{|0\rangle_A, |1\rangle_A\}$ and for B is $\{|+\rangle_B, |-\rangle_B\}$ with Schmidt coefficients $\{\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\}$. Finally

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{50}$$

- (d) Unfortunately, I don't see an easy way to group terms, so I will resort to calculating the singular value decomposition of the matrices. The matrix we look to decompose is

$$a = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{51}$$

which encodes the state $\frac{1}{\sqrt{3}}(|00\rangle + |01\rangle + |10\rangle)$ in the computational basis. Calculating the decomposition in Matlab, I found

$$a = \underbrace{\begin{pmatrix} 0.8507 & 0.5257 \\ 0.5257 & -0.8507 \end{pmatrix}}_{u}\underbrace{\begin{pmatrix} 0.9342 & 0 \\ 0 & 0.3568 \end{pmatrix}}_{d}\underbrace{\begin{pmatrix} 0.8507 & 0.5257 \\ -0.5257 & 0.8507 \end{pmatrix}}_{v} \tag{52}$$

We then define the basis for system A by $|i\rangle_A = \sum_j u_{ji}|j\rangle_A$. This results in $\{0.8507|0\rangle_A + 0.5257|1\rangle_A, 0.5257|0\rangle_A - 0.8507|1\rangle_A\}$. We define the basis for B by $|j\rangle_B = \sum_k v_{ik}|k\rangle_B$, giving $\{0.8507|0\rangle_B + 0.5257|1\rangle_B, -0.5257|0\rangle_B + 0.8507|1\rangle_B\}$. The Schmidt coefficients are just the diagonal entries of d, $\{0.9342, 0.3568\}$. Calculating ρ_A directly (omitting matrix elements which clearly drop out)

$$\rho_A = \mathrm{tr}_B\,\rho = \frac{1}{3}\sum_i {}_B\langle i|\Big(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |00\rangle\langle 10| + |10\rangle\langle 00|\Big)|i\rangle_B$$

$$= \frac{1}{3}\Big(2|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|\Big) \tag{53}$$

- (e) First, we calculate the final state (recall $Y_{-2\pi/3} = I\cos\frac{\pi}{3} + iY\sin\frac{\pi}{3}$)

$$\begin{aligned}
CNOT(H\otimes Y_{-2\pi/3})|00\rangle &= CNOT(H|0\rangle_A \otimes Y_{-2\pi/3}|0\rangle_B) && (54)\\
&= CNOT\Big[\frac{1}{\sqrt{2}}(|0\rangle_A + |1\rangle_A)\otimes(\frac{1}{2}|0\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_B\Big] && (55)\\
&= \frac{1}{\sqrt{2}}|0\rangle_A \otimes \left(\frac{1}{2}|0\rangle_B - \frac{\sqrt{3}}{2}|1\rangle_B\right) + \frac{1}{\sqrt{2}}|1\rangle_A \otimes \left(\frac{1}{2}|1\rangle_B - \frac{\sqrt{3}}{2}|0\rangle_B\right)\\
&= \frac{1}{2\sqrt{2}}\Big[|00\rangle - \sqrt{3}|01\rangle - \sqrt{3}|10\rangle + |11\rangle\Big] && (56)
\end{aligned}$$

Again, we calculate the SVD of

$$B = \frac{1}{2\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \tag{57}$$

which encodes the state in the computational basis. Using Mathematica, we find

$$B = \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}}_{u} \underbrace{\begin{pmatrix} \frac{\sqrt{2+\sqrt{3}}}{2} & 0 \\ 0 & \frac{\sqrt{2-\sqrt{3}}}{2} \end{pmatrix}}_{d} \underbrace{\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}_{v} \tag{58}$$

Following part (d), we see that the basis for A is $\{|-\rangle_A, -|+\rangle_A\}$ and for B is $\{|-\rangle_B, |+\rangle_B\}$ with Schmidt coefficients $\{\frac{\sqrt{2+\sqrt{3}}}{2}, \frac{\sqrt{2-\sqrt{3}}}{2}\}$. Notice that I have been able to eliminate an overall minus sign for the first basis elements, but not for the second set. Lastly, we calculate the reduced density matrix directly to find

$$\begin{aligned}
\rho_A &= \frac{2+\sqrt{3}}{4}|-\rangle\langle-| + \frac{2-\sqrt{3}}{4}|+\rangle\langle+| && (59)\\
&= \frac{1}{2}|0\rangle\langle0| - \frac{\sqrt{3}}{4}|0\rangle\langle1| - \frac{\sqrt{3}}{4}|1\rangle\langle0| + \frac{1}{2}|1\rangle\langle1| && (60)
\end{aligned}$$

6.4 Trace distance

- (a) Let $A = \rho - \sigma$ which is diagonalized as $A = \sum_i \lambda_i |i\rangle\langle i|$, then the trace distance is

$$\begin{aligned}
D(\rho,\sigma) &= \frac{1}{2}\|\rho-\sigma\|_{\text{tr}} = \frac{1}{2}\|A\|_{\text{tr}}\\
&= \frac{1}{2}\,\text{tr}\sqrt{A^\dagger A} = \frac{1}{2}\,\text{tr}\sqrt{\sum_{ij}\lambda_i|i\rangle\underbrace{\langle i|j\rangle}_{\delta_{ij}}\langle j|\lambda_k^*}\\
&= \frac{1}{2}\,\text{tr}\sum_i\sqrt{|\lambda_i|^2}|i\rangle\langle i|\\
&= \frac{1}{2}\sum_i|\lambda_i| && (61)
\end{aligned}$$

- (b) Given the previous result, we find the difference of the two matrices

$$\rho - \sigma = \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| - \frac{2}{3}|+\rangle\langle +| - \frac{1}{3}|-\rangle\langle -| \tag{62}$$

$$= \frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1| - \frac{1}{3}\Big(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\Big)$$
$$- \frac{1}{6}\Big(|0\rangle\langle 0| - |1\rangle\langle 0| - |0\rangle\langle 1| + |1\rangle\langle 1|\Big) \tag{63}$$

$$= \frac{1}{4}|0\rangle\langle 0| - \frac{1}{4}|1\rangle\langle 1| - \frac{1}{6}|0\rangle\langle 1| - \frac{1}{6}|1\rangle\langle 0| \tag{64}$$

$$= \begin{pmatrix} \frac{1}{4} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{4} \end{pmatrix} \tag{65}$$

  The eigenvalues are $\pm\sqrt{13}/12$ so that the trace distance is $\sqrt{13}/12$.

- (c) Define $\psi = \sin\theta/2|e_0\rangle + \cos\theta/2|e_1\rangle$ and $\phi = -\sin\theta/2|e_0\rangle + \cos\theta/2|e_1\rangle$. Again, we first calculate the difference of the density matrices.

$$|\psi\rangle\langle\psi| - |\phi\rangle\langle\phi| = \sin^2\frac{\theta}{2}|e_0\rangle\langle e_0| + \frac{1}{2}\sin\theta|e_0\rangle\langle e_1| + \frac{1}{2}\sin\theta|e_0\rangle\langle e_1| + \cos^2\frac{\theta}{2}|e_1\rangle\langle e_1| \tag{66}$$

$$- \sin^2\frac{\theta}{2}|e_0\rangle\langle e_0| + \frac{1}{2}\sin\theta|e_0\rangle\langle e_1| + \frac{1}{2}\sin\theta|e_0\rangle\langle e_1| - \cos^2\frac{\theta}{2}|e_1\rangle\langle e_1| \tag{67}$$

$$= \sin\theta\Big(|e_0\rangle\langle e_1| + |e_1\rangle\langle e_0|\Big) \tag{68}$$

  The eigenvalues are $\pm\sin\theta$, giving a trace distance of $\sin\theta$.

- (d) Using the definitions above, we have

$$\|\psi - \phi\|^2 = \|2\sin\frac{\theta}{2}|e_0\rangle\|^2 = 4\sin^2\frac{\theta}{2} \tag{69}$$

  Thus $\|\psi - \phi\| = 2\sin\frac{\theta}{2}$ and comparing to part (b), we find that $\sin\theta \leq 2\sin\frac{\theta}{2}$, indicating $D(|\psi\rangle\langle\psi|, |\phi\rangle\langle\phi|) \leq \|\phi - \psi\|$.

- (e) Using our examples, consider $\theta = \pi$, in which case the two states have different phases, e.g $\psi = |0\rangle$, $\phi = -|0\rangle$. Under the quantum norm, we have $\||0\rangle - (-|0\rangle)\| = 2$. However, the trace norm is 0. Thus, the quantum norm for the difference of states (which is not necessarily a quantum state), picks up the non-physical phase difference, while the trace norm does not.

6.5 Double Amplitude

The amplitude channel $\mathcal{A}$, with $p = 1 - e^{-t/T_1}$, has Krause operators

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad A_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \tag{70}$$

- (a) First, we calculate the action of the channel on $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.

$$\mathcal{A}(|+\rangle\langle+|) = \frac{1}{2}\sum_i A_i \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} A_i^\dagger \tag{71}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix} \tag{72}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 1 \\ \sqrt{1-p} & \sqrt{1-p} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \sqrt{p} & \sqrt{p} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix} \tag{73}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & \sqrt{1-p} \\ \sqrt{1-p} & 1-p \end{pmatrix} + \frac{1}{2}\begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \tag{74}$$

$$= \frac{1}{2}\begin{pmatrix} 1+p & \sqrt{1-p} \\ \sqrt{1-p} & 1-p \end{pmatrix} \tag{75}$$

Since the initial state is a pure state, the fidelity (squared) is just

$$F^2(|+\rangle\langle+|, \mathcal{A}(|+\rangle\langle+|)) = \langle+|\frac{1}{2}\begin{pmatrix} 1+p & \sqrt{1-p} \\ \sqrt{1-p} & 1-p \end{pmatrix}|+\rangle \tag{76}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1+p & \sqrt{1-p} \\ \sqrt{1-p} & 1-p \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{77}$$

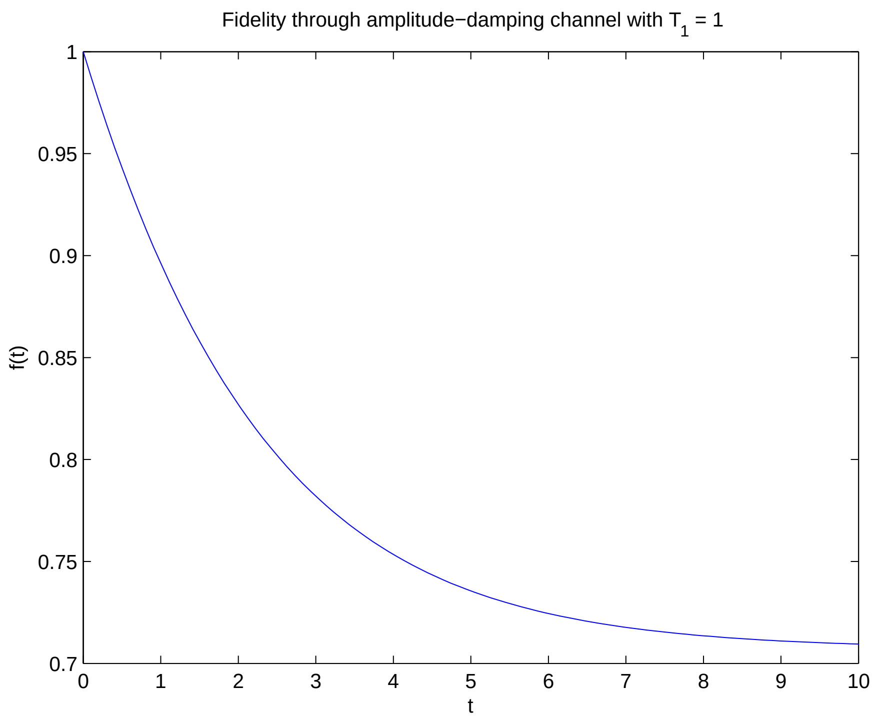
$$= \frac{1}{4}\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1+p+\sqrt{1-p} \\ 1-p+\sqrt{1-p} \end{pmatrix} \tag{78}$$

$$= \frac{1}{4}(2 + 2\sqrt{1-p}) \tag{79}$$

$$= \frac{1}{2}(1 + \sqrt{1-p}) \tag{80}$$

$$= \frac{1}{2}(1 + e^{-\frac{t}{2T_1}}) \tag{81}$$

Thus, $f(t) = \sqrt{F^2} = \frac{1}{\sqrt{2}}\sqrt{1 + e^{-\frac{t}{2T_1}}}$.



We see (and can calculate) that $\lim_{t\to\infty} f(t) = \frac{1}{\sqrt{2}}$.

- (b) We know that a general qubit state can be written as $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$. The amplitude-damping channel changes the state to

$$\mathcal{A}(|\psi\rangle\langle\psi|) = \frac{1}{2}\sum_i A_i \begin{pmatrix} \cos^2\frac{\theta}{2} & e^{-i\phi}\frac{\sin\theta}{2} \\ e^{i\phi}\frac{\sin\theta}{2} & \sin^2\frac{\theta}{2} \end{pmatrix} A_i^\dagger \tag{82}$$

$$= \begin{pmatrix} \frac{1}{2}(1+p+(1-p)\cos\theta) & \frac{1}{2}e^{-i\phi}\sqrt{1-p}\sin\theta \\ \frac{1}{2}e^{i\phi}\sqrt{1-p}\sin\theta & \frac{1}{2}(p-1)(\cos\theta-1) \end{pmatrix} \tag{83}$$

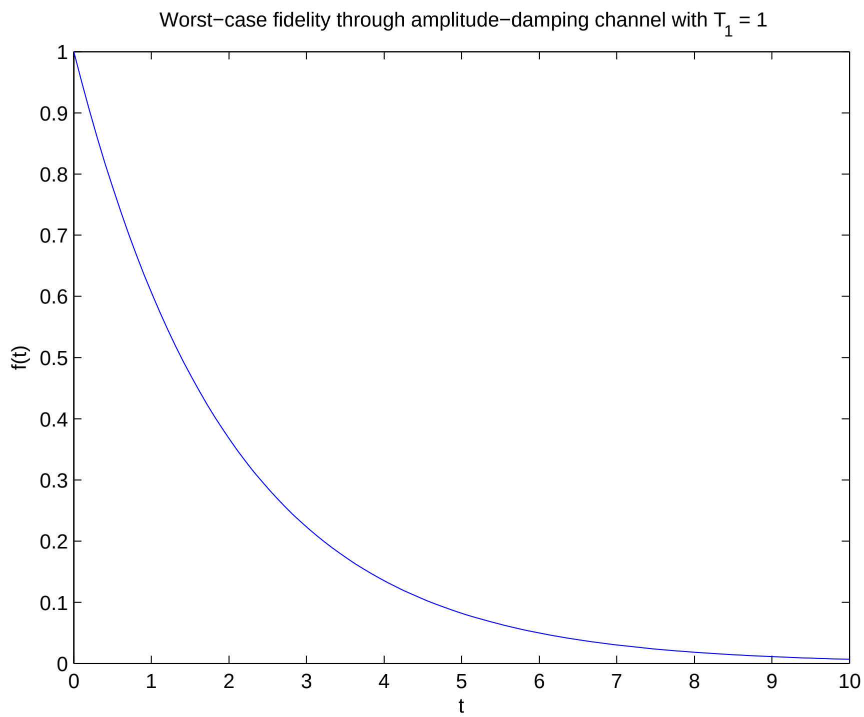
Again, the fidelity for an initial pure state is just $|\langle\psi|\mathcal{A}(|\psi\rangle\langle\psi|)|\psi\rangle|$, which gives

$$F(|\psi\rangle\langle\psi|, \mathcal{A}(|\psi\rangle\langle\psi|)) = \sqrt{\frac{1}{4}\left(3+\sqrt{1-p}-p+2p\cos\theta-\left(-1+\sqrt{1-p}+p\right)\cos 2\theta\right)} \tag{84}$$

To minimize, we take the derivative with respect to θ and solve when the derivative is zero. There are several roots, but checking the second derivative indicates that the minimum occurs for $\theta_m = \pi$ with the minimizing initial state $|\psi_m\rangle = |1\rangle$. Plugging back in, we have

$$F(|\psi_m\rangle\langle\psi_m|, \mathcal{A}(|\psi_m\rangle\langle\psi+m|)) = (1-p)^{1/2} = e^{-t/(2T_1)} \tag{85}$$

In the limit $t\to\infty$, the exponential decays to 0, so that $f(t)$ also goes to 0. This should corroborate the discussion in class, where we noted that this channel pushes the state to $|0\rangle$, or pointing up on the Bloch sphere. Consequently, the fidelity minimizing initial state is the orthogonal $|1\rangle$ state.



- (c) Let $|\phi\rangle = \cos\frac{\theta}{2}|\bar{0}\rangle + e^{i\phi}\sin\frac{\theta}{2}|\bar{1}\rangle$. The action of the amplitude-damping channel is given by

$$\mathcal{A}\otimes\mathcal{A}|\phi\rangle\langle\phi| = \sum_{ij}(A_i\otimes A_j)|\phi\rangle\langle\phi|(A_i\otimes A_j)^\dagger = \sum_{ij} B_{ij}|\phi\rangle\langle\phi|B_{ij}^\dagger \tag{86}$$

where $B_{ij} = A_i\otimes A_j$. There are four such combinations to worry about, which we work through step by step.

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$$B_{00}|\phi\rangle\langle\phi|B_{00}^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} & \frac{e^{i\phi}}{2}\sin\theta & 0 \\ 0 & \frac{e^{i\phi}}{2}\sin\theta & \sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-p} & 0 & 0 \\ 0 & 0 & \sqrt{1-p} & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix} \tag{87}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (1-p)\cos^2\frac{\theta}{2} & \frac{1}{2}e^{i\phi}(1-p)\sin\theta & 0 \\ 0 & \frac{1}{2}e^{i\phi}(1-p)\sin\theta & (1-p)\sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{88}$$

$$= (1-p)|\phi\rangle\langle\phi| \tag{89}$$

–

$$B_{01}|\phi\rangle\langle\phi|B_{01}^\dagger = \begin{pmatrix} 0 & \sqrt{p} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-p}\sqrt{p} \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} & \frac{e^{i\phi}}{2}\sin\theta & 0 \\ 0 & \frac{e^{i\phi}}{2}\sin\theta & \sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{p} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1-p}\sqrt{p} & 0 \end{pmatrix} \tag{90}$$

$$= \begin{pmatrix} p\cos^2\frac{\theta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{91}$$

$$B_{10}|\phi\rangle\langle\phi|B_{10}^\dagger = \begin{pmatrix} 0 & 0 & \sqrt{p} & 0 \\ 0 & 0 & 0 & \sqrt{1-p}\sqrt{p} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} & \frac{e^{i\phi}}{2}\sin\theta & 0 \\ 0 & \frac{e^{i\phi}}{2}\sin\theta & \sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\times \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{p} & 0 & 0 & 0 \\ 0 & \sqrt{1-p}\sqrt{p} & 0 & 0 \end{pmatrix} \tag{92}$$

$$= \begin{pmatrix} p\sin^2\frac{\theta}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{93}$$

–

$$B_{11}|\phi\rangle\langle\phi|B_{11}^\dagger = \begin{pmatrix} 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2\frac{\theta}{2} & \frac{e^{i\phi}}{2}\sin\theta & 0 \\ 0 & \frac{e^{i\phi}}{2}\sin\theta & \sin^2\frac{\theta}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}$$

$$= 0 \tag{94}$$

Summing over these gives a final result

$$\mathcal{A}\otimes\mathcal{A}(|\phi\rangle\langle\phi|) = (1-p)|\phi\rangle\langle\phi| + p\cos^2\frac{\theta}{2}|00\rangle\langle 00| + p\sin^2\frac{\theta}{2}|00\rangle\langle 00| \tag{95}$$

$$= (1-p)|\phi\rangle\langle\phi| + p|00\rangle\langle 00| \tag{96}$$

We now verify that this is the same as apply the E_i operators to $|\psi\rangle$.

$$\begin{aligned}\sum_i E_i|\psi\rangle\langle\psi|E_i^\dagger &= (1-p)|\psi\rangle\langle\psi| \\ &\quad + p|00\rangle\langle 01| (\alpha|01\rangle + \beta|10\rangle) (\alpha^*\langle 01| + \beta^*\langle 10|) |01\rangle\langle 00| \\ &\quad + p|00\rangle\langle 10| (\alpha|01\rangle + \beta|10\rangle) (\alpha^*\langle 01| + \beta^*\langle 10|) |10\rangle\langle 00| \quad (97) \\ &= (1-p)|\psi\rangle\langle\psi| + p(|\alpha|^2 + |\beta|^2)|00\rangle\langle 00| \quad (98) \\ &= (1-p)|\psi\rangle\langle\psi| + p|00\rangle\langle 00| \quad (99)\end{aligned}$$

which is the same as the action of $\mathcal{A}\otimes\mathcal{A}$.

- (d) Again, since we start with a pure state, the fidelity is

$$|\langle\phi|\mathcal{A}\otimes\mathcal{A}(|\phi\rangle\langle\phi|)|\phi\rangle| = |\langle\phi|\left[(1-p)|\phi\rangle\langle\phi| + p|00\rangle\langle 00|\right]|\phi\rangle| = \sqrt{1-p} \quad (100)$$

This is independent of the encoded qubit, i.e. this is the fidelity independent of the choice of the angle θ and phase ϕ. This is the same as the worst-case fidelity calculated in part (b).

- (e) If we measure ZZ, the probability of obtaining $|00\rangle$ ($ZZ = +1$) is just the coefficient of $|00\rangle\langle 00|$ above, which is p. If this outcome is not obtained, the state is $|\phi\rangle$, which has outcome $ZZ = -1$. Clearly, this has fidelity 1 with respect to $|\phi\rangle$ independent of time. We call this an error-detecting code because if we obtain the outcome $+1$ when measuring ZZ, we know an error occurred. Similarly an outcome of -1 tells us that no error occurred. However, we do not have the ability to correct; the process of taking an unknown, arbitrary initial $|\phi\rangle$ to $|00\rangle$ is irreversible.

6.6 Extra Credit:High Fidelity

- (a) Letting $\rho = \frac{1}{2}(I + \vec{r}\cdot\vec{\sigma})$

$$\begin{aligned}\det\rho &= \det\left|\frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z\end{pmatrix}\right| \quad (101) \\ &= \frac{1}{4}\left((1-r_z)(1+r_z) - (r_x+ir_y)(r_x-ir_y)\right) \quad (102) \\ &= \frac{1}{4}\left(1 - r_z^2 - r_x^2 - r_y^2\right) \quad (103) \\ &= \frac{1}{4}(1-\vec{r}^2) \quad (104)\end{aligned}$$

which means $|\vec{r}| = \sqrt{1-4\det\rho}$.

- (b) Consider

$$A = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \quad (105)$$

$$\operatorname{tr} A = a + d \quad (106)$$

$$\det A = ad - bc \quad (107)$$

The eigenvalues of the matrix are given by the characteristic equation

$$\det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc = 0 \tag{108}$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc) = 0 \tag{109}$$

$$= \lambda^2 - \mathrm{tr}\, M\lambda + \det A = 0 \tag{110}$$

which indicates that

$$\lambda_\pm = \frac{1}{2}\left[\mathrm{tr}\, A \pm \sqrt{(\mathrm{tr}\, A)^2 - 4\det A}\right] \tag{111}$$

So that $\lambda_+ + \lambda_- = \mathrm{tr}\, A$ and $\lambda_+\lambda_- = \det A$.

- (c) We look to diagonalize $\rho^{1/2}\sigma\rho^{1/2}$, with ρ associated with Bloch vector $\vec{r}$ and σ with $\vec{s}$. We will use the results from part (b) to determine the eigenvalues.

  Recalling from problem set 3 that $(\vec{r} \cdot \vec{\sigma})(\vec{s} \cdot \vec{\sigma}) = (\vec{r} \cdot \vec{s})I + i(\vec{r} \times \vec{s}) \cdot \vec{\sigma}$, we can easily calculate the trace.

$$\mathrm{tr}\, \rho^{1/2}\sigma\rho^{1/2} = \mathrm{tr}\, \rho\sigma \tag{112}$$

$$= \frac{1}{4}\, \mathrm{tr}\, [(I + \vec{r} \cdot \vec{\sigma})(I + \vec{s} \cdot \vec{\sigma}] \tag{113}$$

$$= \frac{1}{4}\, \mathrm{tr}\, [I + \vec{r} \cdot \vec{\sigma} + \vec{s} \cdot \vec{\sigma} + (\vec{r} \cdot \vec{s})I + i(\vec{r} \times \vec{s}) \cdot \vec{\sigma}]$$

$$= \frac{1}{2}\, [1 + (\vec{r} \cdot \vec{s})] \tag{114}$$

  where we have also use the fact the the Pauli matrices are traceless. Turning to the determinant, we have that $\det(AB) = \det(A)\det(B)$, which allows us to rearrange $\det(\rho^{1/2}\sigma\rho^{1/2}) = \det(\rho^{1/2})\det(\sigma)\det(\rho^{1/2}) = \det(\rho)\det(\sigma)$. Then

$$\det\rho = \det\left|\frac{1}{2}\begin{pmatrix} 1 + r_z & r_x - ir_y \\ r_x + ir_y & 1 - r_z \end{pmatrix}\right| \tag{115}$$

$$= \frac{1}{4}\left(1 - r_z^2 - r_x^2 - r_y^2\right) \tag{116}$$

$$= \frac{1}{4}(1 - \vec{r}^2) \tag{117}$$

  so that $\det\sigma = \frac{1}{4}(1 - \vec{s}^2)$. Thus,

$$\det(\rho^{1/2}\sigma\rho^{1/2}) = \frac{1}{16}(1 - \vec{r}^2)(1 - \vec{s})^2 \tag{118}$$

  Let $\lambda_\pm$ represent the eigenvalues of $\rho^{1/2}\sigma\rho^{1/2}$, then

$$F^2(\rho, \sigma) = \left(\mathrm{tr}\, \sqrt{\rho^{1/2}\sigma\rho^{1/2}}\right)^2 \tag{119}$$

$$= \left(\sqrt{\lambda_+} + \sqrt{\lambda_-}\right)^2 \tag{120}$$

$$= \lambda_+ + \lambda_- + 2\sqrt{\lambda_+\lambda_-} \tag{121}$$

$$= \mathrm{tr}(\rho^{1/2}\sigma\rho^{1/2}) + 2\sqrt{\det(\rho^{1/2}\sigma\rho^{1/2})} \tag{122}$$

$$= \frac{1}{2}\left[1 + (\vec{r} \cdot \vec{s}) + (1 - \vec{r}^2)(1 - \vec{s}^2)\right] \tag{123}$$

where in going to (120), we have taken the trace in the diagonal basis.

- (d) As stated in the hint, we want to show that $ABBA$ and $BAAB$ have the same eigenvalues. Suppose that $ABBA$ has eigenvalue λ_i associated with eigenvector $|i\rangle$. Then,

$$\lambda_i(BA|i\rangle) = BA(\lambda_i|i\rangle) \tag{124}$$
$$= BA(ABBA|i\rangle) \tag{125}$$
$$= (BAAB)(BA|i\rangle) \tag{126}$$

We see that $BAAB$ then has the same eigenvalue λ_i, but with eigenvector $BA|i\rangle$. Thus

$$\|AB\|_{\text{tr}} = \|BA\|_{\text{tr}}$$
$$\text{tr}\sqrt{BAAB} = \text{tr}\sqrt{ABBA} \tag{127}$$

since the trace can be taken in the diagonal basis, in which case it is just the sum of the eigenvalues.

6.7 Extra credit: Miscellaneous channel problems

- (a) First, note that

$$X_a Z_b = \sum_{i,j=0}^{2^n-1} (-1)^{j\cdot b}|i\oplus a\rangle\langle i|j\rangle|j\rangle = \sum_{i=0}^{2^n-1}(-1)^{i\cdot b}|i\oplus a\rangle\langle i| \tag{128}$$

so that an arbitrary matrix element $|l\rangle\langle m|$ is transformed as

$$\sum_{a,b=0}^{2^n-1} X_a Z_b|l\rangle\langle m|(X_a Z_b)^\dagger = \sum_{a,b,i,j=0}^{2^n-1}(-1)^{i\cdot b}|i\oplus a\rangle\langle i|l\rangle\langle m|j\rangle\langle j\oplus a|(-1)^{j\cdot b} \tag{129}$$
$$= \sum_{a,b=0}^{2^n-1}(-1)^{(l\oplus m)\cdot b}|l\oplus a\rangle\langle m\oplus a| \tag{130}$$
$$= \sum_{a}^{2^n-1}\delta_{l,m}|l\oplus a\rangle\langle m\oplus a| \tag{131}$$
$$\tag{132}$$

Now, given that $\rho = \sum_{lm}\alpha_{lm}|l\rangle\langle m|$, we see that

$$\sum_{a,b=0}^{2^n-1}\frac{1}{2^{2n}}X_a Z_b\rho(X_a Z_b)^\dagger = \sum_{a,l,m=0}^{2^n-1}\frac{1}{2^{2n}}\alpha_{l,m}\delta_{l,m}|l\oplus a\rangle\langle m\oplus a| \tag{133}$$
$$= \sum_{a,l=0}^{2^n-1}\frac{1}{2^{2n}}\alpha_{l,l}|l\oplus a\rangle\langle l\oplus a| \tag{134}$$
$$= \sum_a\frac{1}{2^n}|a\rangle\langle a| = \frac{1}{2^n}I \tag{135}$$

where in going to the last line, we note that sum over l just cycles through the diagonal elements out of order and since $\sum_l \alpha_{ll} = 1$, we get an overall factor of 2^n from this sum.

- (b) Again, a general pure state is written is $|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}$. Using Mathematica, we can calculate the action of the Krause operators to find

$$\rho_{\text{out}} = \sum_i A_i|\psi\rangle\langle\psi|A_i^\dagger = \begin{pmatrix} \frac{9}{154}(13+3\cos\theta) & -\frac{9}{22}e^{-i\phi}\sin\theta \\ -\frac{9}{22}e^{-i\phi}\sin\theta & \frac{1}{154}(37-27\cos\theta) \end{pmatrix} \tag{136}$$

If the output state is pure, then $\rho_{\text{out}}^2 = \rho_{\text{out}}$. Again, in Mathematica, we find

$$\rho_{\text{out}}^2 = \begin{pmatrix} \frac{81\left((13+3\cos\theta)^2+49e^{-2i\phi}\sin^2\theta\right)}{23716} & -\frac{9}{22}e^{-i\phi}\sin\theta \\ -\frac{9}{22}e^{-i\phi}\sin\theta & \frac{(37-27\cos\theta)^2+3969e^{-2i\phi}\sin^2\theta}{23716} \end{pmatrix} \tag{137}$$

Given that the off-diagonals are unchanged, we can set $\phi = 0$. Solving either diagonal pair, indicates that $\theta_\pm = \pm\cos^{-1}\left(\frac{1}{3}\right)$. Plugging into our general equation for ψ, we see that the input pure/output pure state pairs are

$$\psi_{\theta^\pm} = \frac{1}{\sqrt{3}}\left(|0\rangle \pm \sqrt{2}|1\rangle\right) \mapsto \frac{1}{\sqrt{11}}\left(3|0\rangle + \sqrt{2}|1\rangle\right) \tag{138}$$

So two different input pure states are taken to the same output pure state.

- (c) We can generally write two distinct non-orthogonal pure states as $|\psi_1\rangle$ and $|\psi_2\rangle = |\psi_1\rangle + |\psi_1^\perp\rangle$. Under the map $\mathcal{E}$, each state is preserved, i.e. $\mathcal{E}(|\psi_1\rangle) = |\psi_1\rangle$ and $\mathcal{E}(|\psi_2\rangle) = |\psi_2\rangle$. But using our definitions

$$\mathcal{E}(|\psi_2\rangle) = \mathcal{E}(|\psi_1\rangle) + \mathcal{E}(|\psi_1^\perp\rangle) \tag{139}$$

$$|\psi_2\rangle = |\psi_1\rangle + \mathcal{E}(|\psi_1^\perp\rangle) \tag{140}$$

which requires $\mathcal{E}|\psi_1^\perp\rangle = |\psi_1^\perp\rangle$. Restricting attention to the two-dimensional subspace spanned by $|\psi_1\rangle$ and $|\psi_1^\perp\rangle$, we immediately see that the only quantum operation which results in the above relations is the identity map on this subspace.

- (d) The phase-damping channel Krause operators are

$$B_0 = \sqrt{1-\frac{p}{2}}I \qquad B_1 = \sqrt{\frac{p}{2}}Z = \sqrt{\frac{p}{2}}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{141}$$

and the amplitude-damping channel Krause operators are

$$A_0 = |0\rangle\langle 0| + \sqrt{1-p'}|1\rangle\langle 1| \qquad A_1 = \sqrt{p'}|0\rangle\langle 1| \tag{142}$$

Transforming under the phase-damping channel, followed by the amplitude channel

$$\mathcal{A}(\mathcal{B}(\rho)) = \sum_{i,j} A_i B_j \rho B_j^\dagger A_i^\dagger \tag{143}$$

requires calculating each $A_i B_j$:

1.

$$A_0 B_0 = \left(|0\rangle\langle 0| + \sqrt{1-p'}|1\rangle\langle 1|\right)\sqrt{1-\frac{p}{2}}I \tag{144}$$

$$= \sqrt{1-\frac{p}{2}}\left(|0\rangle\langle 0| + \sqrt{1-p'}|1\rangle\langle 1|\right) \tag{145}$$

2.

$$A_0 B_1 = \left(|0\rangle\langle 0| + \sqrt{1-p'}|1\rangle\langle 1|\right)\sqrt{\frac{p}{2}}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{146}$$

$$= \sqrt{\frac{p}{2}}\left(|0\rangle\langle 0| - \sqrt{1-p'}|1\rangle\langle 1|\right) \tag{147}$$

3.

$$A_1 B_0 = \sqrt{p'}|0\rangle\langle 1|\sqrt{1-\frac{p}{2}}I \tag{148}$$

$$= \sqrt{p'}\sqrt{1-\frac{p}{2}}|0\rangle\langle 1| \tag{149}$$

4.

$$A_1 B_1 = \sqrt{p'}|0\rangle\langle 1|\sqrt{\frac{p}{2}}\left(|0\rangle\langle 0| - |1\rangle\langle 1|\right) \tag{150}$$

$$= -\sqrt{p'}\sqrt{\frac{p}{2}}|0\rangle\langle 1| \tag{151}$$

We see that A_1B_0 and A_1B_1 are proportional to $|0\rangle\langle 1|$ and that

$$(A_1B_0)^\dagger(A_1B_0) + (A_1B_1)^\dagger(A_1B_1) = p'(1-\frac{p}{2})|1\rangle\langle 1| + p'\frac{p}{2}|1\rangle\langle 1| = p'|1\rangle\langle 1| \tag{152}$$

An equivalent 3 component Krause map is then $\{A_0B_0, A_0B_1, \sqrt{p'}|0\rangle\langle 1|\}$.