A Brief Review of Everything

or

Noise, Codes, and Circuits

by

Bryan Eastin

Errors Matter

Quantum computing has tremendous power in the absence of errors...

...that is, quantum computing has tremendous power in the absence of reality.

To actually make a quantum computer we must first learn to deal with quantum errors.

Single Qubit Pure State Errors

An arbitrary error on a single qubit pure state, $|\psi\rangle = a|0\rangle + b|1\rangle$, can be written as

$$
\begin{aligned}
|\psi\rangle \otimes |0\rangle_E = (a|0\rangle + b|1\rangle) \otimes |0\rangle_E \rightarrow \\
a(|0\rangle \otimes |e_{00}\rangle_E + |1\rangle \otimes |e_{01}\rangle_E) + b(|0\rangle \otimes |e_{10}\rangle_E + |1\rangle \otimes |e_{11}\rangle_E) \\
= (a|0\rangle + b|1\rangle) \otimes \frac{1}{2}(|e_{00}\rangle_E + |e_{11}\rangle_E) \\
+ (a|1\rangle + b|0\rangle) \otimes \frac{1}{2}(|e_{01}\rangle_E + |e_{10}\rangle_E) \\
+ (a|1\rangle - b|0\rangle) \otimes \frac{1}{2}(|e_{01}\rangle_E - |e_{10}\rangle_E) \\
+ (a|0\rangle - b|1\rangle) \otimes \frac{1}{2}(|e_{00}\rangle_E + |e_{11}\rangle_E) \\
= I|\psi\rangle \otimes |e_I\rangle_E + X|\psi\rangle \otimes |e_X\rangle_E + Y|\psi\rangle \otimes |e_Y\rangle_E + Z|\psi\rangle \otimes |e_Z\rangle_E
\end{aligned}
$$

Errors on Pure States

Likewise, errors on an n-qubit pure state can be written in the form

$$|\psi\rangle \otimes |0\rangle_E \to \sum_a E_a|\psi\rangle \otimes |e_a\rangle_E \quad where\ E_a \in \{I, X, Y, Z\}^{\otimes n}$$

Note: The $|e_a\rangle_E$ are not generally normalized or orthogonal.

Nonetheless, we can repair errors if we can determine which Pauli string was applied.

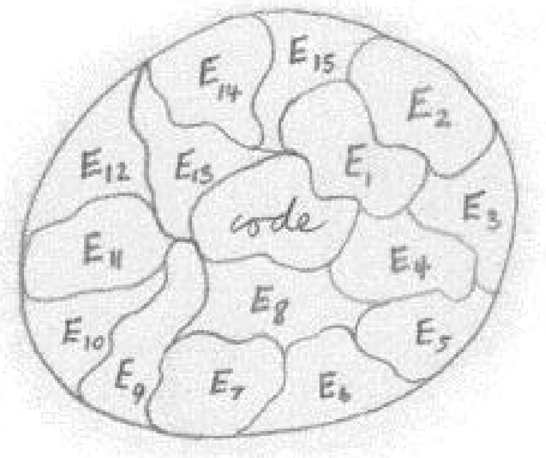
Collective Measurement

Collective measurement can determine the error without damaging the data.

Consider the bit flip code, $|\bar{0}\rangle = |000\rangle$ and $|\bar{1}\rangle = |111\rangle$.

$$X_1(a|\bar{0}\rangle + b|\bar{1}\rangle) = a|100\rangle + b|011\rangle$$

Measuring Z_1Z_2 and Z_2Z_3 yeilds $Z_1Z_2 = -1$ and $Z_2Z_3 = 1$. This informs us that the first qubit was flipped without damaging our data.



What about other kinds of errors?

7 Qubit Code

Let C be the set of strings orthogonal to the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Define

$$|\bar{0}\rangle = \frac{1}{\sqrt{8}} \sum_{w \in C} |w\rangle$$

$$|\bar{1}\rangle = \frac{1}{\sqrt{8}} \sum_{w \in C} |w + 1110000\rangle$$

We can correct bit flips in the same way as before.

To correct phase flips, apply a Hadmard gate to each qubit.

Note that

$$\sum_{w \in C} (-1)^{w \cdot y} = \begin{cases} 2^3 & y \in C^{\perp} \\ 0 & y \notin C^{\perp} \end{cases}$$

$$H^{(7)}|\bar{0}\rangle = \frac{1}{32} \sum_{y} \sum_{w \in C} (-1)^{w \cdot y} |w\rangle$$

$$= \frac{1}{4} \sum_{y \in C^{\perp}} |y\rangle$$

7 Qubit Code Continued

$$H^{(7)}|\bar{1}\rangle = \frac{1}{32}\sum_{y}\sum_{w\in C}(-1)^{w\cdot y+\bar{1}\cdot y}|w\rangle$$
$$= \frac{1}{4}\sum_{y\in C^{\perp}}(-1)^{\bar{1}\cdot y}|y\rangle$$

Since $C^{\perp}$ is the set of strings perpendicular to the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

we again have a code to correct one bit flip.

Correcting a bit flip in this basis corresponds to correcting phase flips since

$$H^{(7)}Z_i|\bar{0}\rangle = H^{(7)}Z_iH^{(7)}H^{(7)}|\bar{0}\rangle$$
$$= X_iH^{(7)}|\bar{0}\rangle$$

All CSS (Calderbaker Shor Steane) Codes rely on the same idea, correct the bit flips, switch to the Hadamard tranformed basis, correct phase flips.

A Brief Note on Distance

Definitions

Weight - Number of non-identity Pauli operators in a tensor product

Distance - The minimum weight Pauli string capable of transforming one codeword into another.

An $[[n, k, d]]$ quantum code

- requires n physical qubits
- encodes k logical qubits
- has a distance of d
- detects Pauli errors of weight $d - 1$
- corrects Pauli errors of weight $\frac{(d-1)}{2}$

Five Qubit Code

Define a $[[5,1,3]]$ code s.t.

- all codewords are $+1$ eigen-states of

$$M_1 = XZZXI$$
$$M_2 = IXZZX$$
$$M_3 = XIXZZ$$
$$M_4 = ZXIXZ$$

- logical X and Z are given by

$$\bar{X} = XXXXX$$
$$\bar{Z} = ZZZZZ$$

Let $S = \{M_i\}^4$,

$$|\bar{0}\rangle = \sum_{K \in S} K|00000\rangle$$
$$|\bar{1}\rangle = \bar{X}|\bar{0}\rangle = \sum_{K \in S} K|11111\rangle$$

All Pauli strings of $weight \leq 2$ anticommute with a stabilizer generator, so d=3.

Recipe for Stabilizers

Summary:

Choose a commuting set of stabilizer generators, $S = \{M_i\}$.

Choose logical operators, $\bar{X}$ and $\bar{Z}$, s.t. $[\bar{X}, M_i] = [\bar{Z}, M_i] = 0$, $\{\bar{X}, \bar{Z}\} = 0$, and $\bar{X}^2 = \bar{Z}^2 = I$.

Define codewords as $+1$ eigenstates of the stabilizers.

Find a codeword, $|\phi\rangle$, s.t. $\bar{Z}|\phi\rangle = |\phi\rangle$.

Define $|\bar{0}\rangle = \sum_{M \in S} M|\phi\rangle$ and $|\bar{1}\rangle = \sum_{M \in S} M\bar{X}|\phi\rangle$

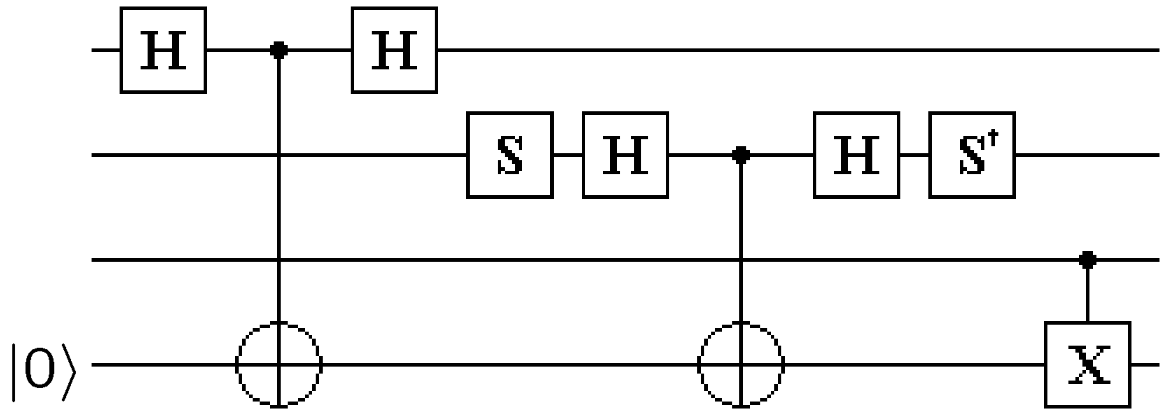
Properties:

$distance = minimum\ commuting\ error\ weight$

$\#\ of\ stabilizer\ generators = n - k$

Measuring Stabilizers

Stabilizers can be measured most directly using modified $C-NOT$ gates. The following circuit shows how to measure $X_1 Y_2 Z_3$.



The operation corresponding to the measurement circuit is

$$E_a|\bar{\psi}\rangle \otimes |0\rangle \rightarrow \begin{cases} E_a|\bar{\psi}\rangle \otimes |0\rangle & if \ [M_i, E_a] = 1 \\ E_a|\bar{\psi}\rangle \otimes |1\rangle & if \ [M_i, E_a] \neq 1 \end{cases}$$

We will discuss later why this is not actually the preferred way to measure stabilizers.

Generalized Error Correction

A general error operator acting on a generic state has the form

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \quad where \sum_i E_i E_i^\dagger \leq 1$$

A complete syndrome circuit implements the unitary

$$U : (E_a|\bar{\psi}\rangle) \otimes |0\rangle \rightarrow (E_a|\bar{\psi}\rangle) \otimes |a\rangle$$

This unitary enables the correction of arbitrary errors on a mixed state.

$$\sum_i E_i \rho E_i^\dagger \otimes |0\rangle\langle 0| \xrightarrow{U} \sum_i E_i \rho E_i^\dagger \otimes |i\rangle\langle i| \xrightarrow{measure} E_i \rho E_i^\dagger \otimes |i\rangle\langle i| \xrightarrow{correct} \rho$$

References

- Nielsen, M.A. & Chuang, I.L., Quantum Computation and Quantum Information, Cambridge University Press, 2000

- Preskill,J, From his online compilation at: http://www.theory.caltech.edu/people/preskill/ph229