Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #11 Due Thursday, December 8
(20 points) (at lecture)

11.2 (10 points) Challenge problem. **Group-velocity approximation.** Suppose we have a wave packet traveling to the right,

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, \tilde{f}_R(k') e^{i[k'x - \omega(k')t]} \ .$$

The difference from our previous consideration of waves in one spatial dimension is that we now allow the angular frequency to be a nonlinear function, $\omega(k')$, of the wave number k' (the reason for a putting a prime on the wave number will become clear below); another way of putting this is that the phase velocity, ω/k', is not constant. This leads to a phenomenon called *dispersion*, in which a wave packet spreads over time. We explore such dispersion or spreading in this problem.

The functional form of the angular frequency as a function of wave number, i.e., the function $\omega(k')$, is called a *dispersion relation.* A dispersive wave packet is *not* a solution of the wave equation, but it is a solution of a related equation that we will not explore here. Dispersive wave propagation occurs in many physical situations, not least in the propagation of the wave function of a free particle in nonrelativistic quantum mechanics, where $\hbar\omega = E = p^2/2m = \hbar^2 k'^2/2m$, i.e., $\omega = \hbar k'^2/2m$ and $\omega/k' = \hbar k'/2m = p/2m$. Another prominent example is the propagation of electromagnetic waves in materials with an index of refraction that depends on frequency.

There is one niggling point that has to be confronted every time you deal with Fourier transforms: if $f(x,t)$ is real, then $\tilde{f}_R(-k') = \tilde{f}_R^*(k')$. Since we don't want to be worrying about what is happening at negative wave numbers, we will set $\tilde{f}_R(k')$ equal to zero for negative k'. This means we have a complex wave packet, which works for quantum mechanics; should you be in a situation where you need a real wave packet, you just take the real part.

Consider now a situation where $\tilde{f}_R(k')$ has substantial support only over a narrow interval of wave numbers centered at large wave number k_0. We assume that the width $\Delta k \ll k_0$ of the narrow interval is so small that the phase velocity, ω/k', doesn't vary much across the interval. This allows us to expand $\omega(k')$ in a Taylor series about the center of the interval:

$$\omega(k') = \omega_0 + v_g(k' - k_0) + \frac{1}{2}\alpha(k' - k_0)^2 + (\text{higher-order terms}) \ ,$$

where

$$\omega_0 = \omega(k_0) \ ,$$

$$v_g = \left.\frac{d\omega}{dk'}\right|_{k'=k_0} \ ,$$

$$\alpha = \left.\frac{d^2\omega}{dk'^2}\right|_{k'=k_0} \ .$$

We shall see that the linear term describes a wave packet that moves to the right with the *group velocity* v_g. The quadratic and higher-order terms describe spreading, or dispersion, of the wave packet.

Before going further, it is useful to separate the wave packet into the "carrier wave," which is the rapid oscillation at the central wave number k_0, and an envelope that defines the wave packet and within which the carrier wave does its rapid oscillations. To do this, we introduce a new integration variable, $k = k' - k_0$, which has its zero at the center of the wave packet:

$$f(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{f}_R(k_0+k) e^{i[(k_0+k)x - \omega(k_0+k)t]} = e^{ik_0x} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{F}(k) e^{i[kx-\omega(k_0+k)t]} \ , \tag{1}$$

The function $\tilde{F}(k) = \tilde{f}_R(k_0+k)$, which is peaked at $k=0$, is the Fourier transform of the envelope function $F(x)$, i.e.,

$$F(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \tilde{F}(k) e^{ikx} \ , \qquad \tilde{F}(k) = \int_{-\infty}^{\infty} dx\, F(x) e^{-ikx} \ .$$

To summarize, we have a wave given by Eq. (1), which is a rapid oscillation at the high wave number k_0 within an envelope $F(x)$. The Fourier transform $\tilde{F}(k)$ of the envelope has support only within a narrow interval Δk around $k=0$. The dispersion relation has been expanded to quadratic order, giving

$$\omega(k_0+k) = \omega_0 + v_g k + \frac{1}{2}\alpha k^2 + (\text{higher-order terms}) \ . \tag{2}$$

Now let's get going.

(a) *Show* that if one retains only the constant and linear terms in the dispersion relation (2), the wave packet consists of "carrier wave" with wave number k_0, which moves to the right with the phase velocity $v_p = \omega_0/k_0$, and an envelope that moves to the right with the group velocity v_g. This is called the *group-velocity approximation.*

No matter how nonlinear the dispersion relation is, we can always consider a wave packet that is narrow enough in k that we can justify keeping only the constant and linear term in the Taylor expansion (1). This means, however, in accordance with our discussion of isolated wave packets, that the wave packet has an extent $a \gtrsim 2\pi/\Delta k$ in position. If we are forced to make Δk very small to justify keeping only the linear term, the wave packet will become very extended in position. Certainly not all wave packets are of this sort, and thus not all wave packets can be treated in the group-velocity approximation. To see what happens outside the group-velocity approximation, let's go one further order in the expansion (1) and also keep the quadratic term.

(b) Assume now that the envelope is a normalized Gaussian,

$$F(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2} \ ,$$

centered at the origin. This Gaussian spreads over a spatial extent of a few times σ. *Find* the wave packet $f(x,t)$ if one retains all three terms in the dispersion relation (2). *Explain* what your result means for the spreading (or dispersion) of the wave packet.