Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #1 Due Tuesday, August 30
(60 points) (at lecture)

1.6 (10 points) Challenge problem. A point charge q of mass m moves in the x-y plane under the influence of a uniform magnetic field pointing normal to that plane in the $-z$ direction, i.e., $\mathbf{B} = -B\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the z axis. The velocity $\mathbf{v}(t) = v_x(t)\hat{\mathbf{x}} + v_y(t)\hat{\mathbf{y}}$ of the charge obeys Newton's equation

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v}\times\mathbf{B} = qB\hat{\mathbf{z}}\times\mathbf{v}$$

(this is the SI equation; in the more natural cgs Gaussian units, the right-hand side would have a $1/c$, where c is the speed of light).

(a) *Show* that the two components of the velocity vector, v_x and v_y, both of which are real, obey the equations

$$\frac{dv_x}{dt} = -\omega v_y \qquad \text{and} \qquad \frac{dv_y}{dt} = \omega v_x \ ,$$

where $\omega = qB/m$. The motion described by these equations is called *cyclotron motion*, and ω is called the *cyclotron frequency*.

(b) Consider now the complex combination $z = v_x + iv_y$. *Show* that z obeys the simple equation

$$\frac{dz}{dt} = i\omega z$$

and that the general solution of this equation is $z(t) = z(0)e^{i\omega t}$.

(c) By taking the real and imaginary parts of the above solution, *show* that the solution of Newton's equation of part (a) is

$$v_x(t) = v_x(0)\cos\omega t - v_y(0)\sin\omega t \ ,$$
$$v_y(t) = v_x(0)\sin\omega t + v_y(0)\cos\omega t \ .$$

(d) Let $w = x + iy$ the the complex number that contains the position of the point charge as its real and imaginary parts. *Find* the general solution for $w(t)$. The general solution describes circular motion about a point $u = u_x + iu_y$ in the complex plane. *Find* u in terms of the initial conditions $z(0)$ and $w(0)$; write your solution for $w(t)$ in terms of u. *Translate* the solution into solutions for $x(t)$ and $y(t)$.

Now let's return to Newton's equation, and do the same problem using vectors.

(e) *Show* that

$$\mathbf{v}(t) = R(t)\mathbf{v}(0) = \mathbf{v}(0)\cos\omega t + \hat{\mathbf{z}}\times\mathbf{v}(0)\sin\omega t \ , \qquad R(t) = \cos(\omega t)\, I + \sin(\omega t)\,\hat{\mathbf{z}}\times \ ,$$

is the general solution of Newton's equation of part (a). *Write* this solution in terms of the matrix that describes rotations in the x-y plane.

(f) *Integrate* the solution of part (e) to find the solution for the charge's position vector $\mathbf{r}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$, assuming that the center of the circular motion is the origin. *Draw* the trajectory of the charge, and assuming that the charge is initially on the positive x axis, *indicate* on your drawing the velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t) = d\mathbf{v}/dt$ at $\omega t = 0, \pi/2, \pi$, and $3\pi/2$.