Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #3 Due Thursday, September 15
(50 points) (at lecture)

3.1 (10 points) Let R_1 be the rotation operator for a rotation by $\pi/4$ about the z axis; this operator changes the Cartesian basis vectors according to

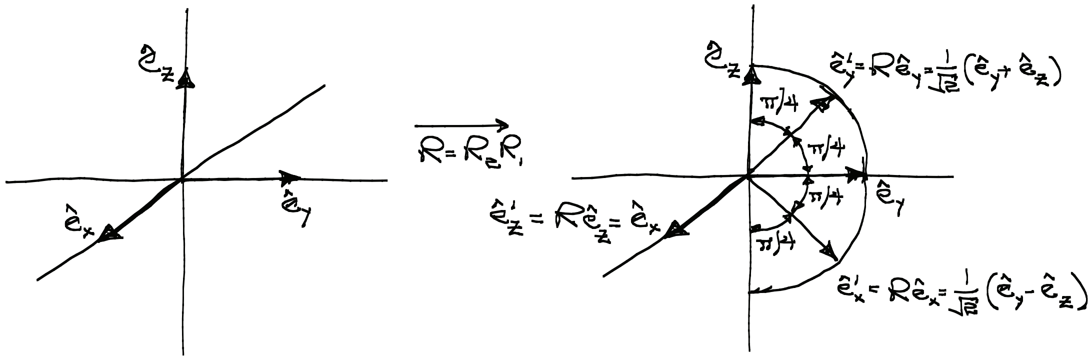
$$\begin{aligned} R_1\hat{\mathbf{e}}_x &= \frac{1}{\sqrt{2}}(\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\ , \\ R_1\hat{\mathbf{e}}_y &= \frac{1}{\sqrt{2}}(-\hat{\mathbf{e}}_x + \hat{\mathbf{e}}_y)\ , \\ R_1\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z\ . \end{aligned}$$

Let R_2 be the rotation operator for a rotation by $\pi/2$ about the y axis; this operator changes the Cartesian basis vectors according to

$$\begin{aligned} R_2\hat{\mathbf{e}}_x &= -\hat{\mathbf{e}}_z\ , \\ R_2\hat{\mathbf{e}}_y &= \hat{\mathbf{e}}_y\ , \\ R_2\hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_x\ . \end{aligned}$$

This entire problem is quite easy and is just to make you get used to thinking about both active and passive transformations.

(a) *Show* that the composite rotation $R = R_2R_1$ rotates the original basis vectors to new, primed basis vectors as shown in the drawing below.



(b) Now consider the vector $\mathbf{A} = 2\hat{\mathbf{e}}_x - \hat{\mathbf{e}}_y$. *Show* that under the composite rotation R (this is an active transformation), $\mathbf{A}$ rotates to

$$\mathbf{A}' = R\mathbf{A} = \frac{1}{\sqrt{2}}\big(\hat{\mathbf{e}}_y - 3\,\hat{\mathbf{e}}_z\big)\ .$$

(c) *Find* the matrix elements $R_{jk} = \hat{\mathbf{e}}_j \cdot R\hat{\mathbf{e}}_k$ of the composite rotation, and use these to *show* that the components of $\mathbf{A}'$ in the original basis are those found in part (b).

(d) Use the matrix elements R_{jk} to *find* the components of $\mathbf{A}$ in the rotated (primed) basis (this is a passive transformation).

3.2 (10 points) A particle moves along the trajectory $\mathbf{r}(t) = b\,\hat{\mathbf{e}}_x + vt\,\hat{\mathbf{e}}_y$, where b and v are constants. All your answers should be written in terms of b, v, t, and the appropriate unit basis vectors.

(a) *Write* the particle's velocity $\mathbf{v}(t)$ in Cartesian coördinates. *Describe* in words the magnitude and direction of the velocity. Is the particle accelerating?

(b) *Write* the particle's velocity $\mathbf{v}(t)$ in cylindrical coördinates.

(c) The particle's angular momentum is $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$, where m is the particle's mass. *Find* $\mathbf{L}$, and *write* it in both Cartesian and cylindrical coördinates.

3.3 (10 points) A particle moves in a plane elliptical orbit described by the position vector

$$\mathbf{r}(t) = 2b\cos\omega t\,\hat{\mathbf{e}}_x + b\sin\omega t\,\hat{\mathbf{e}}_y\ .$$

(a) *Draw* the particle's trajectory. Include in your drawing the position vector at times $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, $t = 3\pi/4\omega$, and $t = \pi/\omega$.

(b) *Find* $\mathbf{v}$ and the particle's speed v as functions of t, and include in your drawing the velocity $\mathbf{v}$ at the times listed in part (a). At what point(s) in the trajectory is the speed greatest?

(c) *Find* $\mathbf{a}$ as a function of t. *Describe* the direction of $\mathbf{a}$. At what point(s) in the trajectory is the magnitude of the acceleration greatest?

(d) What is the angle between $\mathbf{v}$ and $\mathbf{a}$ at the times listed in part (a)?

3.4 (10 points) Let M be a 3×3 matrix with matrix elements M_{jk}. The determinant of the matrix, written in terms of the antisymmetric symbol and using the summation convention, is

$$\det M = \epsilon_{jkl}M_{1j}M_{2k}M_{3l}\ . \tag{1}$$

If you are unsure how this works, it would be a good idea to write it out explicitly to convince yourself that it works.

If we permute the rows of a matrix, the determinant changes by the *sign* of the permutation; i.e., it remains the same for an even permutation and changes sign for an odd permutation. Thus the determinant can also be written as

$$\epsilon_{mnp}\det M = \epsilon_{jkl}M_{mj}M_{nk}M_{pl}\ . \tag{2}$$

In this form, m, n, and p can take on any value, but are not summed over. Equation (1) is a special case of Eq. (2), obtained from Eq. (2) by setting $mnp = 123$.

(a) Use Eq. (2) to *show* that the determinant is also given by

$$\det M = \frac{1}{3!}\epsilon_{mnp}\epsilon_{jkl}M_{mj}M_{nk}M_{pl}\ .$$

Explain why the $3! = 6$ appears in the result.

(b) Use Eq. (2) to *show* that the inverse of M is given by

$$(M^{-1})_{jk} = \frac{1}{\det M}\frac{1}{2}\epsilon_{jlm}\epsilon_{knp}M_{nl}M_{pm} \; ; \tag{3}$$

we require, of course, that $\det M \neq 0$ so that the inverse exists. [Hint: You will need to recall that the matrix elements of the product of M and N are given by $(MN)_{jk} = M_{jl}N_{lk}$; i.e., these are the inner product of the jth row of M with the kth column of N.]

The quantities

$$C_{jk} = \frac{1}{2}\epsilon_{jnp}\epsilon_{klm}M_{nl}M_{pm}$$

are called co-factors; the matrix $C = ||C_{jk}||$ whose matrix elements are the co-factors is called, not surprisingly, the matrix of co-factors. Its transpose, C^T, is called the adjugate matrix. Equation (3) says that

$$(M^{-1})_{jk} = \frac{1}{\det M}C_{kj} = \frac{1}{\det M}(C^T)_{jk} \qquad \Longleftrightarrow \qquad M^{-1} = \frac{C^T}{\det M} \; ;$$

i.e., M^{-1} is the adjugate matrix divided by $\det M$.

(c) *Verify* directly that Eq. (3) gives the inverse of M.

3.5 (10 points) Challenge problem