Phys 366 Mathematical Methods of Physics Fall 2015

Homework Assignment #4 Due Thursday, September 17
(50 points) (at lecture)

4.1 (10 points) The height of a hill in meters is given by

$$h(x, y) = 2xy - 3x^2 - 4y^2 - 18x + 28y + 12 ,$$

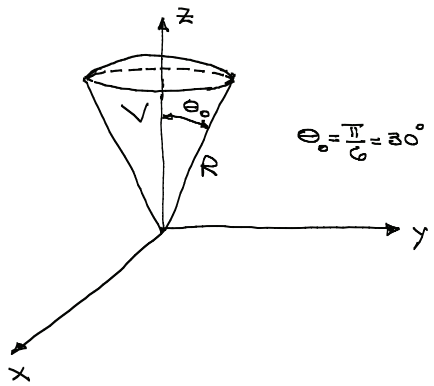
where x is the distance in kilometers east of the origin and y is the distance in kilometers north of origin.

(a) Where is the top of the hill, and how high is it?

(b) How steep is the hill at $x = y = 1$, i.e., what is the slope in the direction of steepest ascent? In which compass direction is the slope at $x = y = 1$ steepest?

4.2 (10 points) Calculate the gradient and the Laplacian of $1/r$, where r is the spherical radial coördinate, at all points such that $r \neq 0$. What is the flux of $\nabla(1/r)$ through a sphere of radius a centered at the origin? How do you reconcile your results with the divergence theorem?

4.3 (10 points) Consider the "ice-cream-cone" volume V shown below and the surface S that encloses it.



The sides of the volume are a cone with opening angle $\theta_0 = \pi/6$ (30°), and the top of the cone is a piece of a sphere of radius R. For the vector field

$$\mathbf{v} = r^2 \sin\theta\, \hat{\mathbf{e}}_r + 4r^2 \cos\theta\, \hat{\mathbf{e}}_\theta + r \tan\theta\, \hat{\mathbf{e}}_\phi ,$$

evaluate both

$$\int_V \nabla \cdot \mathbf{v}\, d\tau \qquad \text{and} \qquad \oint_S \mathbf{v} \cdot d\mathbf{a} .$$

You should find that the two integrals are equal, thereby verifying the divergence theorem for this case.

4.4 (10 points) **The divergence theorem and vector integrals.**

(a) Let f be a scalar field and $\mathbf{F}$ a vector field. *Show* that

$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F} \, .$$

(Hint: Work in Cartesian coördinates.)

(b) If f is a differentiable scalar field, *show* that

$$\int_V \nabla f \, d\tau = \oint_S f \, d\mathbf{a} \, ,$$

where V is an arbitrary volume, S is the closed surface that bounds V, and $d\mathbf{a}$ is the outward-directed area element on S. [Hint: Consider the component of the vector volume integral along an arbitrary (constant) unit vector $\hat{\mathbf{e}}$, and use the result of part (a) to convert the volume integral into a form suitable for application of the divergence theorem.]

(c) *Evaluate* the vector surface integral

$$\oint_S \mathbf{A} \cdot \mathbf{r} \, d\mathbf{a}$$

over a spherical shell S of radius b, centered at the origin. In the integral, $\mathbf{A}$ is a *constant* vector,

$$\mathbf{r} = r\hat{\mathbf{e}}_r = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y + z\hat{\mathbf{e}}_z$$

is the *position vector* from the origin to a point in three-dimensional space ($\mathbf{r}$ varies as you integrate over the shell), and $d\mathbf{a}$ is the outward-directed area element on S. [Hint: Use the result of part (b). Think about the level surfaces (contours) defined by the function $\mathbf{A} \cdot \mathbf{r}$.]

4.5 (10 points) Challenge problem