Phys 366 Mathematical Methods of Physics Fall 2016

Homework Assignment #5 Due Friday, September 30
(40 points) (at lecture)

5.1 (10 points) Let $\mathbf{A}$ be the vector field defined by

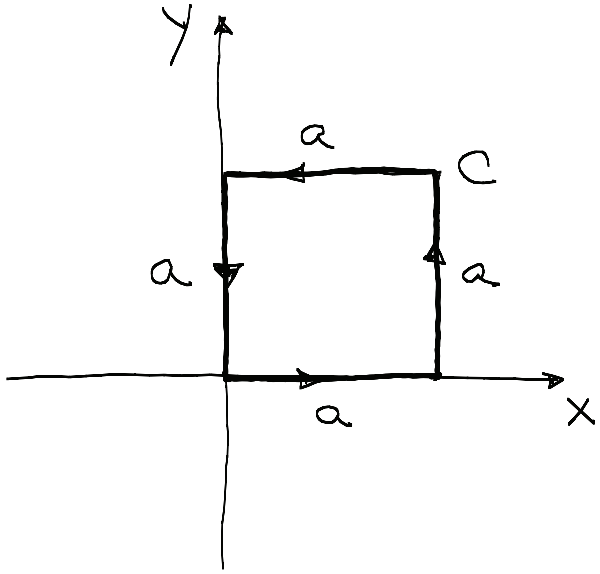
$$\mathbf{A} = y^2\hat{\mathbf{e}}_x - x^2\hat{\mathbf{e}}_y$$

in Cartesian coördinates.

(a) *Evaluate* the line integral

$$\oint_C \mathbf{A} \cdot d\ell$$

around the closed curve C shown in the drawing below. The curve C is a square with sides of length a, lying in the x-y plane, and the line integral should be done in the direction indicated on C.

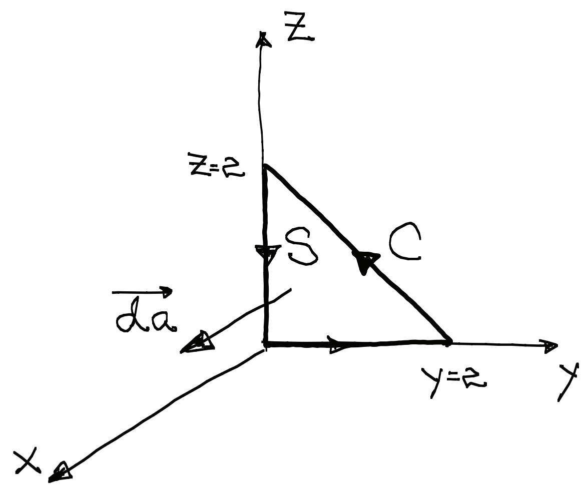


(b) *Evaluate* the same line integral by using Stokes's theorem to convert the line integral to a surface integral.

5.2 (10 points) Consider the vector field $\mathbf{v} = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3zx\hat{\mathbf{z}}$. *Evaluate* both

$$\int_S \nabla \times \mathbf{v} \cdot d\mathbf{a} \qquad \text{and} \qquad \oint_C \mathbf{v} \cdot d\ell$$

for the surface S and bounding curve C shown in the drawing below.



5.3 (10 points) **The divergence theorem and vector integrals**

(a) Let $\mathbf{F}$ and $\mathbf{G}$ be vector fields. *Show* that

$$\nabla \cdot (\mathbf{F}\times\mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}) \; .$$

(Hint: Work in Cartesian coördinates. Use the antisymmetric symbol.)

(b) If $\mathbf{A}$ is a differentiable vector field, *show* that

$$\int_V \nabla \times \mathbf{A} \, d\tau = \oint_S d\mathbf{a}\times\mathbf{A} \; ,$$

where V is an arbitrary volume, S is the closed surface that bounds V, and $d\mathbf{a}$ is the outward-directed area element on S. [Hint: Consider the component of the volume integral along an arbitrary (constant) unit vector $\hat{\mathbf{e}}$, and use the result of part (a) to convert the volume integral into a form suitable for application of the divergence theorem.]

(c) *Evaluate* the vector surface integral

$$\oint_S \rho \, \hat{\mathbf{e}}_\phi \times d\mathbf{a}$$

over a spherical shell S of radius R, centered at the origin. In the integral ρ is the *cylindrical* radial coordinate, $\hat{\mathbf{e}}_\phi$ is the azimuthal basis vector in cylindrical coördinates, and $d\mathbf{a}$ is the outward-directed area element on S. Your answer should be in terms of R, and you must give the direction of the vector. In this part use the result of part (b) to convert the surface integral into a volume integral.

(d) *Evaluate* the surface integral in part (c) directly, instead of by converting it to a volume integral.

5.4 (10 points) Challenge problem